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EDITORIAL COMMITTEE
H. C. CARVER
A. L. O'TOOLE
T. E. RAIFORD

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APPLICATIONS OF TWO OSCULATORY FORMULAS

By John L. Roberts

INTRODUCTION

The main purpose of this paper is to illustrate how Mr. Jenkins' osculatory formulas[1] (A) and (B) can be applied in a convenient manner. The first section of this paper will be little more than a summary of some of the formulas contained in the other three articles. The second section will contain the applications.

I. SOME MATHEMATICS OF THE FORMULAS

The Woolhouse notation will in this paper be used to stand for the differences of u_{x+n} which represents the given values of a function. The general formulas are

$$y_x = y_0 + x\Delta y_0 + \tfrac{1}{2}x(x-1)B + \tfrac{1}{6}x(x-1)(x-\tfrac{1}{2})C; \qquad (1)$$

and

$$y_x = u_0 + xa_1 + \tfrac{1}{2}x(x-1)B + \tfrac{1}{6}x(x-1)(x-\tfrac{1}{2})C. \qquad (2)$$

The special formulas belonging to (2) are

$$B = b - \tfrac{1}{6}d \text{ and } C = c_1 - \tfrac{1}{6}e_1\,, \qquad (A)$$

where b and d are defined by $b = \tfrac{1}{2}(b_0 + b_1)$ and by $d = \tfrac{1}{2}(d_0 + d_1)$; and

$$B = b \text{ and } C = 0. \qquad (B)$$

The special formulas belonging to (1) are

$$y_0 = u_0 + \tfrac{1}{4}b_0\,, \qquad B = b, \qquad \text{and } C = 0; \qquad (C)$$

and

$$y_0 = u_0 - \tfrac{1}{36}d_0\,, \qquad B = b - \tfrac{1}{6}d, \qquad \text{and } C = c_1 - \tfrac{1}{6}e_1\,. \qquad (D)$$

Formula (C) is equivalent to Mr. Jenkins' formula (A). Also (D) is equivalent to his formula (B).

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[1] This paper presupposes a knowledge of three other articles. The first one by Mr. Wilmer A. Jenkins is entitled "Graduation Based on a Modification of Osculatory Interpolation," and is printed in the October 1927 issue of the Transactions of the Actuarial Society of America. The other two papers are mine. One of them is entitled "Some Practical Interpolation Formulas," and is printed in the September 1935 issue of these Annals. The other one entitled "A Family of Osculatory Formulas" is printed in the October 1935 issue of the Transactions.

II. APPLICATIONS OF (C) AND (D)

First, there is the problem of selecting suitable examples to which (C) and (D) can be applied. Secondly, we will then apply in a convenient manner the formulas to these examples.

The problem of selecting suitable examples will now be considered. "The non-reproducing characteristic of" formula (D) "raises the question of what will happen in the graduation of a series whose fourth differences are all positive, say. The answer is that the graduated series will lie everywhere below the observed points and that the observations will not be correctly represented by the interpolated series." On the other hand, if we select a series whose fourth differences change frequently in sign, (D) because of its non-reproducing characteristic has valuable smoothing possibilities. In like manner, (C) may be valuable when the second differences change frequently in sign. Mr. Jenkins gives at quinquennial ages rates of mortality which were graphically determined from the published American Men Ultimate Experience. Since the fourth differences of these rates change frequently in sign, we will apply (D) to a few of these rates. So far as I know no suitable actuarial examples have been found to which (C) can be applied. However, there is the possibility that (C) might be valuable in some sciences. Since I do not know of any suitable real example to which (C) can be applied, we will apply it to a trivial series whose second differences change frequently in sign.

We are now ready to apply in a convenient manner (C) and (D) to the examples selected in the preceding paragraph.

First, we will apply (C). I have in my other article applied (B) in a convenient manner. This method with little change can be applied to (C). If it is desired to apply (C) at either end of the table where values of u_x are not available for the calculation of the second differences, it can be assumed they vanish. It is convenient if S and S^2 represent respectively the major differences Δu_x and $\Delta^2 u_x$ in such a manner that they are arranged centrally in the working illustration. It is convenient if s and s^2 represent respectively the minor differences δy_x and $\delta^2 y_x$. The quantity y_0 can be computed by $y_0 = u_0 + \frac{1}{4}b_0$, and y_1 can be computed in like manner. Since we wish in the working illustration of (C) to interpolate four values between y_0 and y_1, the middle $s = \delta y_{.4} = .2\Delta y_0$, and $s^2 = .04B = .02(b_0 + b_1)$. We can by the use of the foregoing method apply (C) to suitable functions, whose given values can be represented by $f(r)$. Then, it follows from the definition of u_x that $f(r) = u_x$. It might prevent confusion if it is stated that x and r are related to each other in such a way that we always interpolate between y_0 and y_1. We shall now apply (C) to the case when $f(r)$ represents the trivial series shown at top of page 3.

Finally, we will apply (D). Mr. Henderson has applied (A) in a very convenient manner. His method with little change can be applied to (D). If it is desired to apply (D) at either end of the table where values of u_x are not available for the calculation of the differences required, it can be assumed that the fourth differences that can not be computed vanish, and the required

r	$f(r)$	S	S^2	y_x	s	s^2
0	1		0	1.0	.18	
1				1.18	.14	
2		1		1.32	.10	−.04
3				1.42	.06	
4				1.48	.02	
5	2		−2	1.5		
6				1.5		
7		−1		1.5	.00	.00
8				1.5		
9				1.5		
10	1		2	1.5	.02	
11				1.52	.06	
12		1		1.58	.10	.04
13				1.63	.14	
14				1.82	.18	
15	2		0	2.0		

differences can be filled in consistently with that assumption. It is convenient if S, S^2, and S^3 represent respectively the major differences Δu_x, $\Delta^2 u_x$, and $\Delta^3 u_x$ in such a manner that they are arranged centrally in the working illustration. It is convenient if s, s^2, and s^3 represent the minor differences so that by definition $s = s_x = \delta y_x$, $s^2 = s_x^2 = \delta^2 y_{x-.2}$, and $s^3 = \delta^3 y_x$. The first $s^2 = \delta^2 y_{-.2} = .04(b_0 - \frac{1}{6}d_0)$. The last $s^2 = \delta^2 y_{.8} = .04(b_1 - \frac{1}{6}d_1)$. The quantity y_0 can be computed by $y_0 = u_0 - \frac{1}{3\,6}d_0$, and y_1 can be computed in like manner. The middle $s = \delta y_{.4} = .2\Delta y_0 - s^3$. We are now in position to apply (D) to the quinquennial rates of mortality.

Age	Rate	S	S^2	S^3	s^4
72	.07010				
		.03808			
77	.10818		.00861		
		.04669		.01799	
82	.15487		.02660		−.03745
		.07329		−.01946	
87	.22816		.00714		.14518
		.08043		.12572	
92	.30859		.13286		.00000
		.21329		.12572	
97	.52188		.25858		.00000

Age	y_x	s	s^2	s^3
82	.15591	12612	.001314	
83	.168522	13527	915	
84	.182049	.014043	516	− .000399
85	.196092	14160	117	
86	.210252	13878	− .000282	
87	.22413	13460	− .000682	
88	.237590	13977	.000517	
89	.251567	.015693	1716	.001199
90	.267260	18608	2915	
91	.285868	22722	4114	
92	.30859	28006	.005314	
93	.336596	34326	6320	
94	.370922	.041652	7326	.001006
95	.412574	49984	8332	
96	.462558	59322	9338	
97	.52188		.010343	

SOME SIMPLE DEVELOPMENTS IN THE USE OF THE COEFFICIENT OF STABILITY

By C. H. Forsyth

Some time ago the writer proposed[1] a coefficient of stability C_s to be used to measure the stability of a statistical series, where that coefficient is defined by the relation

$$C_s = \frac{\sigma^2}{M} \tag{1}$$

where M denotes the arithmetic mean and σ^2 the square of the dispersion of the terms of the series. It was proposed to regard series as unstable (Lexian) for which the value of the coefficient exceeded unity, and stable otherwise. The only essential way in which such a procedure differs *in results* from the traditional method is that it includes as stable those series for which the value of the coefficient lies between unity and q the probability of failure of the event under investigation—series which would be classed as unstable according to the traditional method. Stable series—according to either standard—are found so rarely in practice and therefore so many series are accepted as fairly stable which come anywhere near meeting the requirements that replacing q by unity as the line of demarcation affects the classification of no known series but adds to the effectiveness of the avowed purpose and use of the proposed coefficient—to avoid the round-about work of computing values of probabilities. Another merit of the use of the coefficient is that it enables one to measure and therefore *compare* the stability of several series—a feature which we shall illustrate later.

In brief, such a coefficient provides a means of introducing the whole Lexian theory into Federal publications such as those on vital statistics, since a comparison of the values of the coefficient for, say different communities or countries, would be readily grasped by any reader, whereas the traditional method would prove too subtle and laborious, and allow no ready comparison of results.

For purpose of orientation let us illustrate the situation by analyzing a simple series both ways—the traditional way and by the use of the coefficient of stability. As an example, let us consider the death rates of white infants under one year of age for 1919 (considered on page 89 of the Handbook) for those states whose frequencies of births are comparable or which vary little from

[1] Journal of the American Statistical Association, June, 1932.

their average of 47,830—where the number of deaths for each state has been adjusted to this average as a base.

	Adjusted Deaths X	$X - 3659$	$(X - 3659)^2$
Cal.	3350	−309	95481
Conn.	4700	1041	1083681
Ind.	3732	73	5329
Kan.	3253	−406	164836
Ky.	3686	27	729
Minn.	3159	−500	250000
N. Car.	3541	−118	13924
Va.	3732	73	5329
Wis.	3780	121	14641
	$9\overline{)32933}$	1335−1333	$\overline{)1633950}$
	$M = 3659$		$181550 = \sigma^2$
			$\sigma = 426$

The traditional method would be:

The mean $M = np = 3659$ where $n = 47{,}830$.

$$\text{Hence } p = \frac{3659}{47830} \text{ and } q = \frac{44171}{47830}$$

$$\text{and } \sigma_B^2 = npq = 3659\left(\frac{44171}{47830}\right) = 3378$$

$$\text{whence } \sigma_B = 58.15$$

which is the value of the dispersion we should expect if the basic probability were constant throughout. But the value of the dispersion proves to be $\sigma = \sqrt{181550} = 426$, and the comparison of the values shows that the basic probability to be very variable and therefore the series to be very unstable or Lexian.

The computation of the value of the coefficient of stability is much more simple and direct

$$C_s = \frac{\sigma^2}{M} = \frac{181550}{3659} = 49.6$$

whose excess over unity also clearly indicates the instability of the series.

Since proposing the coefficient of stability the writer has been impressed by the overwhelming proportion of existing series (such as birth rates, various kinds of death rates, etc.) which employ arbitrary bases (such as "per thousand," "per ten thousand," etc.) usually without mention of the actual base. It is obvious, of course, that such rates, or occurrences per arbitrary base, say b, can first be adjusted to give occurrences per actual base, say B (assuming that

base B* can be determined) but the work can evidently be performed much easier. For, since the original series (per arbitrary base b) $X_1, X_2, \cdots X_N$ would become, on adjustment, $\frac{B}{b}X_1, \frac{B}{b}X_2, \cdots \frac{B}{b}X_N$, the mean would become $\frac{B}{b}M$ and the square of the dispersion $\left(\frac{B}{b}\sigma\right)^2$, whence the formula for the coefficient of stability would become

$$C_s = \frac{\sigma^2}{M} \cdot \frac{B}{b} \tag{2}$$

As an example, let us consider the general death rates, per 10,000, of New Zealand for the years 1921–30.

	X	$X - 86$	$(X - 86)^2$
1921	87	1	1
1922	88	2	4
1923	90	4	16
1924	83	−3	9
1925	83	−3	9
1926	87	1	1
1927	85	−1	1
1928	85	−1	1
1929	88	2	4
1930	86	0	0
	10)862	10−8	)46
	M = 86.2		4.6

This example illustrates the danger of using the coefficient of stability unless the series consists of actual occurrences or unless the actual base is given due consideration. Without due consideration of the actual base (here the population of New Zealand) one might easily fall into the error of regarding the value of the coefficient of stability as 4.6/86.2 and, therefore, the series as very stable. But the population of New Zealand is about a million and a half and, therefore the true value of the coefficient of stability is

$$C_s = \frac{4.6}{86.2}\,\frac{1{,}500{,}000}{10{,}000} = 8.0$$

* Strictly speaking, this actual base B should be constant throughout the series; otherwise the successive numbers of occurrences—the terms of the series—would not be comparable. Where, however, the base B varies little from term to term—as usually happens even in the best of series, such as a series of some kind of rates of the same community over a short interval—the variation can be ignored, in which case base B (to which the terms of the series are adjusted) usually means the arithmetic mean of the different bases. In the first treated above, the investigation was limited to certain states in an effort to comply with the rule just mentioned but the example is a poor one since the variations are still dangerously too large. The situation is saved by the conclusive results.

which shows the series to be unstable. However, before we condemn New Zealand's death rates too severely, let us compare her record with those of other important countries, including our own, for the same period.

General Death Rates (per 10,000)

	M	C_s
New Zealand	86.2	8
Australia	94.3	90
Sweden	120.4	96
Scotland	137.3	139
Austria	151.1	536
United States	118.0	830
England-Wales	121.3	1117
France	170.3	1129
Spain	193.7	2190
Italy	163.5	2760
Germany	125.4	6040
Japan	206.4	6800

These results show how extremely unstable most series of general death rates are and that the series for New Zealand, while unstable according to our strict criterion, enjoys quite an enviable position practically in a class by itself. Parenthetically, these results also illustrate fairly well the triviality, with respect to results, of replacing q by unity as the critical value of the coefficient of stability, discussed at the beginning of this article.

The values of the coefficient listed above would, of course, be reduced somewhat in most cases if the trend of the series were first eliminated but the writer has gone though all this work and found it not worth while—that is, the series would still remain markedly unstable.

Another development proves useful when, as frequently happens, the actual base B is unknown to a degree of accuracy desirable for use in formula (2).

From the inequality $\frac{\sigma^2}{M}\frac{B}{b} \leqq 1$

we obtain

$$B \leqq \frac{Mb}{\sigma^2} \tag{3}$$

which is to be used to show how small an actual base should be for the given series to be stable. As an example, let us consider the maternal mortality, per 10,000 live births, in the so-called expanding registration area of the United States.

Maternal Deaths in the United States (per 10,000 live births) (Expanding Registration Area)

	X	$X-66$	$(X-66)^2$
1923	67	1	1
1924	66	0	0
1925	65	−1	1
1926	66	0	0
1927	65	−1	1
1928	69	3	9
1929	70	4	16
1930	67	1	1
1931	66	0	0
1932	64	−2	4
	10)665	9−4	)33
	66.5		3.3

Hence, by formula (3), $B \leqq \frac{66.5}{3.3}(10{,}000)$ or about 200,000. The number of live births varies so greatly that we should probably find it impossible to agree upon a satisfactory number[2] to use as an actual base for such an "expanding area" but we should all agree that it would be so much greater than 200,000 that the instability of the series would be unquestioned.

One must be careful in comparing the results of two or more investigations like the one just conducted. For example, the analogous result for Canada for the same period yields $B \leqq 113{,}000$ and we might conclude, too hastily, that the United States series is more stable (or less unstable) whereas any knowledge whatever of the numbers of live births of the two countries would show that Canada comes much closer to fulfilling her requirement than the United States and that the palm must go to Canada. For one thing, Canada has about the population of New York city and New York city has about 100,000 live births annually. In any case, *close* decisions in matters of this kind would be difficult without sufficient information in regard to actual bases.

There is still another situation which is interesting but of much less importance because of the rarity of its occurrence. It will be recalled that the coefficient of stability was devised mainly to avoid the use and computation of probabilities and that the only difference between the results by the traditional method and by the use of the coefficient of stability lies in the trivial replacement of the critical value q by unity. In the traditional method of analysis, but by comparing the value of the coefficient of stability with q, the coefficient is evidently always, strictly speaking, a function of the actual base B. In other words, there is no statistical series, however stable it may seem—except

[2] It was in the neighborhood of two million in 1932.

for the trivial case when all the terms of the series are exactly the same—but what would be unstable if the base were small enough. It is possible to formulate the limit once for all below which the given (otherwise seemingly stable) series would prove unstable.

If, in the relation $\sigma^2 \leqq npq$ (for stability) we replace p by M/n, q by $1 - M/n$ and then n by B, we obtain

$$\sigma^2 \leqq M - \frac{M^2}{B} \text{ or } \frac{M^2}{B} \leqq M - \sigma^2$$

whence, finally

$$B \geqq \frac{M^2}{M - \sigma^2} \tag{4}$$

where the transference of the term $M - \sigma^2$ from one side to the other should cause no apprehension since, by hypothesis, $\sigma^2 < M$ and $M - \sigma^2$ is therefore always positive. We propose to employ formula (4) in those rare cases where the value of the coefficient of stability of actual occurrences—but without reference to an actual base—is less than unity—that is, where the given series proves to be stable according to the method proposed by the writer—and determine the upper limit of the values of the base B for which the series would be unstable according to the traditional method of analysis. As an illustration, let us consider the familiar series of annual football fatalities in this country for the period 1906–1930* (omitting the years when no records were kept).

Football Fatalities

1906	11	1917	12
1907	11	1921	12
1908	13	1923	18
1909	12	1925	20
1911	11	1926	9
1912	13	1927	17
1913	5	1928	18
1914	13	1929	12
1915	15	1930	13

It is easily verified that $C_s = \frac{11.942}{13.055}$ which is clearly less than unity; whence the series clearly seems stable. Applying formula (4)

$$B \geqq \frac{13.055^2}{13.055 - 11.942} \quad \text{or } 153$$

which shows that the given series is stable as long as the total number of football players exceeds the number 153. A recent news item quoted an estimate of the number players participating in games of four hundred colleges as about

13,000 and over 600,000 including high schools and all. We can then definitely say that the series just considered is stable. Such a conclusion has no bearing, of course, upon what might happen if other terms were added to the series. It happens that adding the records for the next five years—1931(33), 1932(32), 1933(27), 1934(25), 1935(30)—would change the whole series to an unstable one with $C_s = 56.9/16.6 = 3.4$; but, obviously, the additional records belong to a new regime of collection.

INTERNAL AND EXTERNAL MEANS ARISING FROM THE SCALING OF FREQUENCY FUNCTIONS

By Edward L. Dodd

The scaling[1] of frequency functions has been discussed from the standpoint of maximum likelihood. But the likelihood criterion to be satisfied sometimes leads to a minimum likelihood; and sometimes to neither a maximum nor a minimum. Scaling will be studied in this paper with reference to the likelihood actually secured, and also with reference to the character of means obtained, whether internal or external.

SECTION 1. INTRODUCTION

It is well known that a scale obtained in a curve-fitting process is sometimes a mean. Thus, with the normal function

$$\frac{1}{a\sqrt{2\pi}}\, e^{-(x/a)^2/2}, \tag{1}$$

if the scale a is to be obtained from measurements, $x_1, x_2, \cdots, x_n$, we commonly accept the value

$$a = \left\{\frac{1}{n}\sum x_i^2\right\}^{1/2}; \tag{2}$$

that is, the root-mean square of the measurements. Here, the positive value of a is naturally taken. It is called the standard deviation, and thought of as an appropriate new unit of measure.

But even with the x's all negative, and the a taken positive, O. Chisini[2] considered it proper to regard a as a mean of the x's, albeit an *external* mean. From Chisini's viewpoint, this a whether regarded as positive or negative is primarily a solution of

$$x_1^2 + x_2^2 + \cdots + x_n^2 = a^2 + a^2 + \cdots + a^2. \tag{3}$$

In this sum of squares, the single number a may be *substituted* for each of the x's. Perhaps this kind of mean should be called a *substitutive* mean to distinguish it from the means of general analysis which are always internal.

---

[1] Fisher, R. A., "On the mathematical foundation of theoretical statistics," Philosophical Transactions of the Royal Society of London, Series A, Vol. 222, 309–368, (1921). See p. 338.

[2] Chisini, O., "Sul concetto di media," Periodico di matematico, Series 4, Vol. 9, 106–116, (1929).

The normal function is a particular case of a more general function:

$$\text{Constant} \cdot a^{-1} e^{\phi(t)}, \qquad \phi(t) = -t^p/p, \qquad t = x/a. \tag{4}$$

The likelihood method to find the scale a for this function leads to power means, including the arithmetic mean, the root-mean-square, root-mean-cube, etc., for $p = 1, 2, 3$, etc.

The word *scale* will be used only for a positive number,—which then may be regarded as a unit of measurement.

For measurements, $x_1, x_2, \cdots, x_n$ Chisini regarded M as a mean, relative to a function G, provided

$$G(x_1, x_2, \cdots, x_n) = G(M, M, \cdots, M). \tag{5}$$

If a solution of this equation is

$$M = F(x_1, x_2, \cdots, x_n), \tag{6}$$

and c is a possible value for the x's, it follows at once that

$$F(c, c, \cdots, c) = c, \tag{7}$$

or at least one value of this F is c. Conversely, if (7) is satisfied, it is but a change of notation to replace c in (7) by M, and to combine this with (6) to obtain

$$F(x_1, x_2, \cdots, x_n) = F(M, M, \cdots, M). \tag{8}$$

Hence, this F which in (6) gives explicit form to the implicit M found in (5) may also be thought of as a mean-forming function, such as G in (5). Briefly, F is a particular G. Thus $F(x_1, x_2, \cdots x_n)$ is a mean of $x_1, x_2, \cdots, x_n$, if F is so constructed that (7) is satisfied when the arguments are all equal.

Inasmuch as a frequency function $f(t)$ is non-negative, $\log_e f(t)$ is real,—say $\phi(t)$ plus constant. Following R. A. Fisher, it will be convenient to write

$$f(t) = Ca^{-1} e^{\phi(t)}, \qquad C = \text{Constant} \tag{9}$$

With location m already determined, the x's will be thought of as measured from m. And we set

$$t = x/a, \qquad t_i = x_i/a, \qquad i = 1, 2, \cdots, n. \tag{10}$$

The "productive" probability—to yield $x_1, x_2, \cdots, x_n$—is then

$$L = \Pi f(t_i) = C^n a^{-n} e^{\Sigma\phi(t_i)}. \tag{11}$$

This is proportional[3] to the "likelihood" of a. Also—it may be noted in passing—the productive probability is also proportional to the *a posteriori* probability, if a constant *a priori* probability is postulated. The likelihood will here be taken as $\Pi f(t_i)$ itself; and it will be designated by L,—in Fisher's

[3] Loc. Cit., Fisher, p. 310.

notation, $L = \log \Pi$. Of course, Π and $\log \Pi$ take maximum values simultaneously, if at all. From (11) it follows[4] that

$$-a \cdot \partial \log L/\partial a = n + \Sigma t_i \phi'(t_i) = \Sigma\{t_i \phi'(t_i) + 1\}. \tag{12}$$

The equation

$$\Sigma t_i \phi'(t_i) + n = 0 \qquad (i = 1, 2, \cdots, n) \tag{13}$$

will be called the likelihood condition, whether this leads to maximum likelihood, to minimum likelihood, or to neither. A second differention[5] leads to

$$a^2 \cdot \partial^2 \log L/\partial a^2 = \Sigma t_i^2 \phi''(t_i) - n = \Sigma\{t_i^2 \phi''(t_i) - 1\}. \tag{14}$$

When negative, this indicates a maximum likelihood; when positive, a minimum likelihood for the a obtained from (13).

Preparatory to the theorems of the next section, just one more matter will be discussed. The unit for t is arbitrary; and it may be convenient to write, with $k \neq 0$,

$$\phi(t) = \phi(ku) = \Phi(u), \qquad t = ku. \tag{15}$$

Then

$$t\phi'(t) = u\Phi'(u). \tag{16}$$

Suppose, now, that a positive constant k can be found such that $k\phi'(k) = -1$. Then, with $t = ku$, as postulated,

$$1 \cdot \Phi'(1) = k\phi'(k) = -1. \tag{17}$$

Thus $\Phi'(1) = -1$,—or as it will now be written $\phi'(1) = -1$,—is no more restrictive than the condition that some positive k exists such that $k\phi'(k) = -1$.

SECTION 2. GENERAL THEOREMS CONCERNING THE SCALE AS A MEAN

THEOREM I

Given the frequency function

$$f(t) = Ca^{-1} e^{\phi(t)}, \qquad t = x/a, \qquad t_i = x_i/a, \quad C = \text{Constant}. \tag{18}$$

And suppose that

$$\phi'(1) = -1. \tag{19}$$

Suppose, also, that for given $x_1, x_2, \cdots, x_n$, the likelihood condition (13), now written

$$\textstyle\sum_1^n (x_i/a)\phi'(x_i/a) + n = 0, \tag{20}$$

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[4] Loc. Cit., Fisher, p. 338.

[5] Loc. Cit., Fisher, p. 339.

has a positive solution.

$$a = F(x_1, x_2, \cdots, x_n). \tag{21}$$

Then this a, the scale, is a mean.

Proof. With each $x_i = 0$, (20) cannot be satisfied.

But if, with $c \neq 0$, we take each $x_i = c$, and at the same time set $a = c$, then, by (19), $\Sigma = -n$; and thus (20) which gives a implicitly is satisfied. The explicit a in (21) is therefore such a function F that (7) is satisfied. Hence, the scale a is a mean.

THEOREM II

Given the frequency function

$$f(t) = Ca^{-1}e^{\phi(t)}, \qquad t = x/a, \qquad t_i = x_i/a, \quad C = \text{Constant}. \tag{18}$$

Suppose that

$$\phi'(1) = -1, \tag{19}$$

and that

$$|t\phi'(t)| < 1 \quad \text{if} \quad |t| < 1. \tag{22}$$

Moreover, suppose that the likelihood condition (20) for measurements $x_1, x_2, \cdots, x_n$, has a positive solution a. Then

$$a \leqq \text{Maximum } |x_i|. \tag{23}$$

Or, suppose that, in place of (22), we have

$$|t\phi'(t)| > 1 \quad \text{if} \quad |t| > 1; \tag{24}$$

and that $t\phi'(t)$ keeps the same sign, if $|t| > 1$. Then

$$\text{Minimum } |x_i| \leqq a. \tag{25}$$

Proof. Suppose, if possible, that $a > \text{Max } |x_i|$. Then each $|x_i/a| < 1$, and by (22), $|(x_i/a)\phi'(x_i/a)| < 1$. Then (20) is not satisfied, since $|\Sigma| < n$. Thus the hypothesis is contradicted.

Now (25) is satisfied at once if any $x_i = 0$. But suppose, on the other hand, that $\text{Min } |x_i| > 0$; and, if possible, that $a < \text{Min } |x_i|$. Then, by (24) et seq., since $|x_i/a| > 1$, it follows that $|\Sigma| > n$. And thus (20) is again contradicted.

THEOREM III

Given the frequency function

$$f(t) = Ca^{-1}e^{\phi(t)}, \qquad t = x/a, \qquad t_i = x_i/a, \quad C = \text{Constant}; \tag{18}$$

and set $\psi(t) = t\phi'(t) + 1$. Suppose that

$$\lim_{t \to 0} \psi(t) = \alpha, \qquad \lim_{|t| \to \infty} \psi(t) = \beta, \qquad \alpha\beta < 0. \tag{26}$$

And suppose that $\psi(t)$ is continuous when $t \neq 0$.

Then, for any set of real numbers, $x_1, x_2, \cdots, x_n$, of which none is zero, there exists a positive number a, as scale, such that the likelihood condition

$$\sum_1^n (x_i/a)\,\phi'(x_i/a) + n = 0 \tag{20}$$

is satisfied.

The conclusion is also valid, if in place of the limit β, there is postulated

$$\lim_{t\to -b+0} \psi(t) = -\alpha \mid \infty \mid = \lim_{t\to c-0} \psi(t), \tag{27}$$

where $b > 0$, $c > 0$, and $\psi(t)$ is continuous for $-b < t < 0$ and for $0 < t < c$. That is, the new limits are to be infinite with sign opposite to that of α.

Proof. The limits for $t \to 0$ and for $\mid t \mid \to \infty$ are the same as the limits for $a \to \infty$ and $a \to 0+$,—noting that $t = x/a$, $x \neq 0$. Thus $\Sigma\psi(t_i)$ changes sign as a goes from $0+$ to ∞. Hence, since $\psi(t)$ is continuous, (20) is satisfied for some positive a.

For the proof of the second part of the theorem, suppose that $x_n > 0$ and that x_n is the greatest x_i. Then with $a > x_n/c$, but approaching x_n/c, $\psi(x_n/a)$ becomes infinite with sign opposite to that of α. Furthermore, in $\Sigma\psi(x_i/a)$, the positive x's $< x_n$ have a negligible effect; and thus $\lim \Sigma\psi(x_i/a)$, as $a \to (x_n/c) + 0$, is infinite with sign opposite to that of α, when this sum Σ is taken for the positive x's. Likewise, if $x_1 < 0$, and is the least x_i, $\lim \Sigma\psi(x_i/a)$, as $a \to (-x_1/b) + 0$, is infinite with sign opposite to that of α, when this sum is taken for the negative x's. If, now, the measurements happen to be all positive, we think of a as approaching $x_n/c + 0$; and the continuity condition leads to an a which makes $\Sigma\psi(x_i/a) = 0$. Likewise, if the measurements happen to be all negative, we use $-x_1/b + 0$. If both positive and negative x's appear, we use the greater of the two ratios $-x_1/b$ and x_n/c.

SECTION 3. SOME FAIRLY REGULAR FREQUENCY FUNCTIONS

To illustrate the foregoing theorems in a somewhat general manner, consider the measurements, $x_1, x_2, \cdots, x_n$, and with $t = x/a$, $t_i = x_i/a$, set up the function:

$$f(t) = Ca^{-1} \mid kt \mid^p (1 + k^2t^2)^{-q}\, e^{-r \mid kt \mid^s}, \tag{28}$$

where, as before, C is a suitably chosen constant.

Suppose also that

$$p > -1, \qquad q \geqq 0, \qquad r \geqq 0, \qquad s \geqq 0; \tag{29}$$

and that either

$$r > 0,\ s > 0 \qquad \text{or} \qquad r = 0,\ 2q > p + 1. \tag{30}$$

Then with $\phi(t) = \log f(t)$, it follows that, when $t \neq 0$,

$$t\phi'(t) + 1 = (p + 1) - rsk^s \mid t \mid^s - 2qk^2t^2(1 + k^2t^2)^{-1}. \tag{31}$$

Now the condition $1 \cdot \phi'(1) = -1$ would be satisfied if $\Psi(k) = 0$, where

$$\Psi(k) = rsk^{s+2} + rsk^s + (2q - p - 1)k^2 - (p + 1). \tag{32}$$

But, under the conditions (29) and (30) $\Psi(0) < 0$, and $\Psi(\infty) > 0$. Hence, there is a positive k for which $\Psi(k) = 0$. Then if k be assigned this value, (19) is satisfied; and by Theorem I, any scale a that the likelihood condition (20) may lead to is a mean. But, by Theorem III a scale a will actually exist —indeed, for any positive k that may be used in (29); since the limit of $t\phi'(t) + 1$ is positive as $t \to 0$, and is negative as $| t | \to \infty$.

Moreover, if in (29), the further condition $-1 < p \leqq 0$ is introduced, (22) is satisfied. And, thus, $a \leqq$ Maximum $| x_i |$. Also, $| t\phi'(t) |$ increases with $| t |$. Hence, by (24) et seq., Minimum $| x_i | \leqq a$.

If in (28), we set $q = 0$, $s = 1$, $r > 0$, and confine our attention to positive x and t, there is obtained the Pearson Type III. Reference to (32) shows that $\Psi(k) = 0$ if $k = (p + 1)/r$. With this substitution,

$$f(t) = C^1 a^{-1} t^p e^{-(p+1)t}, \qquad C' = \text{Constant}. \tag{33}$$

Since $\phi'(1) = -1$, any solution of the likelihood condition is a mean. Here, with $t > 0$, $t\phi'(t) = p - (p + 1)t$, and $t^2\phi''(t) - 1 = -(p + 1)$. From (14) we see that, with $p + 1 > 0$, any mean obtained corresponds to maximum likelihood and the single maximum found is actually the largest value. Moreover, with the measurements, $x_1, x_2, \cdots, x_n$, all positive, a scale a will exist,—as noted in the general case (28).

In passing, it may be noted that Type III appears[6] rather naturally in a form giving $\phi'(1) = -1$ at once, without any transformation. Here, then, a scale is a mean.

Given the Pearson Type I in the form

$$f(t) = Ca^{-1}(b + kt)^p(c - kt)^q, \quad t = x/a, \quad b > 0, \quad c > 0, \quad | pq | > 0. \tag{34}$$

If $p + q + 1 > 0$, it is possible to find a positive k so that with $\phi = \log f$, $\phi'(1) = -1$. In this case, any scale found by the likelihood condition is a mean. With k thus chosen, $f(t)$ has essentially the same form as it would have if $k = 1$. Hence for convenience, let us simply set $k = 1$ in the above equation. Then for $-b < t < c$,

$$\psi(t) = t\phi'(t) + 1 = 1 + pt(b + t)^{-1} - qt(c - t)^{-1}.$$

Suppose now that $p > 0$ and $q > 0$. Then Theorem III may be applied; since $\lim \psi(t) = 1$, as $t \to 0$; but $\lim \psi(t) \to -\infty$, as $t \to -b + 0$, or as $t \to c - 0$.

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[6] Carver, H. C., Handbook of Mathematical Statistics, Chap. VII, see p. 105, Line 4, noting that $\phi' = y'/y$.

Hence a scale a satisfying the likelihood condition exists. Moreover, the likelihood is at a maximum; since, with $-b < t < c$,

$$t^2\phi''(t) - 1 = -pt^2(b + t)^{-2} - qt^2(c - t)^{-2} - 1 < 0.$$

This maximum is also the largest value for all values of a.

If the Pearson Type IV is given in the form

$$f(t) = Ca^{-1}(1 + k^2t^2)^{-p}\, e^{q \arctan kt}, \qquad t = x/a \tag{35}$$

then if $p > 1/2$, it is possible to find a positive k which will make $\phi'(1) = -1$. In this case, any scale a is a mean. Moreover—for any $k \neq 0$—the limit of $t\phi'(t) + 1$ is 1 for $t \to 0$ and is $1 - 2p$ for $t \to \infty$. Hence, by Theorem III, if $p > 1/2$, as above, then a scale a exists satisfying the likelihood condition (20).

SECTION 4. FREQUENCY FUNCTIONS WITH CERTAIN PECULIARITIES

The theorems of section 2 give sufficient conditions, which in some cases may not be necessary. Nevertheless, by violating certain hypotheses, particular functions may be set up which exhibit various peculiarities.

For the Pearson Types, the differential equation is

$$\frac{y'(t)}{y(t)} = \phi'(t) = \frac{a_0 + a_1t}{b_0 + b_1^t + b_2t^2}, \qquad t = x/a. \tag{36}$$

The determination of a positive scale a by the Fisher likelihood process is impossible here, in case $a_0 = 0$, $a_1 > 0$, $b_0 + b_1t + b_2t^2 > 0$. For in this case $t\phi'(t) \geqq 0$; and thus (20) cannot be satisfied. The U-shaped Type II curves are in this class. Likewise, if $a_0 \neq 0$, $a_1 = 0$, and $b_0 + b_1t + b_2t^2 > 0$,—for example, with $b_2 > 0$, $b_1^2 < 4b_0b_2$,—and the measurements all happen to have the same sign as a_0, such scaling is impossible.

For the purpose of constructing peculiar functions we may take $c > 0$ and require that the measurements x_i be either $-c$ or c—with at least one $-c$ and at least one c—and that $\phi(t)$ be an even function. Then $\phi(-c) = \phi(c)$ and (11) becomes

$$L = [Ca^{-1}\, e^{\phi(c/a)}]^n. \tag{37}$$

The likelihood condition (13) reduces to

$$0 = \psi(t) = t\phi'(t) + 1 = (c/a)\phi'(c/a) + 1, \tag{38}$$

with the right member an even function of c/a. And from (14), a maximum likelihood is indicated when

$$(c/a)^2\phi''(c/a) - 1 < 0, \tag{39}$$

with the left member likewise an even function. A minimum likelihood is indicated if the left member is positive.

Let us apply this to the case where

$$(40) \quad \phi(t) = (-2/3)\log(1 - 3|t|); \qquad t\phi'(t) = 2|t|(1 - 3|t|)^{-1}.$$

The likelihood condition (38) is satisfied only when $t = \pm 1$. Also $\phi'(1) = -1$. Thus the only means are the internal means $\pm c$; and the only scale conformable to (38) is $a = c$. But this has minimum likelihood; since $1 \cdot \phi''(1) - 1 = \frac{1}{2} > 0$. For positive t, this function (40) is a Pearson Type.

Consider next a function of the form (28),—with $p = -1.25$, $q = -0.5$, however,—for which (31) becomes

$$(41) \quad t\phi'(t) + 1 = -1/4 - t^2/4 + t^2/(1 + t^2) = -(1 - t^2)^2/4(1 + t^2).$$

whence $\phi'(1) = -1$, $\phi''(1) = +1$, $\phi'''(1) = -3$. Here the likelihood condition (38) has but a single absolute solution $|t| = 1$, leading to the single scale $a = c$, and to the two internal means, $\pm c$. But, in this case $1 \cdot \phi''(1) - 1 = 0$, so that $\partial^2 \log L/\partial a^2 = 0$. Moreover, for $t = 1$, $\partial^3 \log L/\partial a^3 = a^{-3} \neq 0$. Thus, the only scale obtained by the likelihood method (38)—viz., $a = c$— has a likelihood which is neither at a maximum nor at a minimum.

Another anomalous function is that given by

$$(42) \qquad \phi(t) = t^4 - 2.5t^2, \qquad t = \pm c/a.$$

The likelihood condition (38) leads to

$$\psi(t) = (1 - t^2)(1 - 4t^2) = 0.$$

The only solutions are $t = \pm 1$, giving internal means $\pm c$; and $t = \pm 1/2$, giving external means $\pm 2c$. And from (39) et seq., it can be shown that the internal mean and scale, $a = c$ has minimum likelihood, while the external mean and scale, $a = 2c$, has maximum likelihood.

But it will be noted that a maximum value for a vicinity does not always signify a largest value for the entire possible range. Indeed, for the function (42), $a = 2c$ has maximum likelihood without having the largest likelihood. To avoid such an anomaly, a necessary condition is that as $|t| \to \infty$, $\psi(t) \to -\infty$; as seen by taking the logarithm of L in (37), noting that as $a \to 0$, $(-\log a) \to +\infty$.

Finally avoiding the anomaly just mentioned, let us set up a frequency function, using the $\psi(t)$ in (38), and writing

$$\psi(t) = 1 + t\phi'(t) = (1 - 2t^2)(1 - t^2)(1 - 0.9t^2).$$

From this it follows readily that

$$(43) \qquad \phi(t) = K - 1.95t^2 + 1.175t^4 - 0.3t^6, \qquad K = \text{Constant}.$$

This, with $t_i = \pm c/a$, leads to an internal mean or scale $a = c$ with minimum likelihood, a nearby scale $a = c\sqrt{0.9}$ with maximum likelihood—differing indeed only slightly from the minimum just mentioned—and another scale $a = c\sqrt{2}$ having maximum likelihood, and this likelihood is indeed greater

than that for any other positive value of a. The external mean $a = c\sqrt{2}$ in this case has the largest likelihood. This may be checked by the use of the logarithm of L as it appears in (37), in which the important part is $\phi(c/a) - \log a$.

In passing it may be noted that if $\psi(t)$ has the form $\psi(t) = (1 - t)H(t)$, with $H(1) \neq \infty$, and $t_i = x_i/a$; then any solution a of the likelihood condition $\psi(t) = 0$ is a mean,—by Theorem I.

SECTION 5. SUMMARY

When the R. A. Fisher likelihood method is used to find an "optimum" scale for frequency functions, it sometimes happens that this scale is a well known mean or at least is a *substitutive* mean—See Equation (5). Or a simple transformation (15) may often put the frequency function into such a form. Conditions are given under which a scale will be a mean. Under further conditions this mean will be internal—at least as regards absolute values. Finally, under certain conditions, a scale will exist.

But for certain functions not satisfying these conditions, anomalies appear. The scale given by the usual likelihood condition may be a scale with a minimum likelihood. Sometimes the likelihood will be at neither a maximum nor a minimum. In certain simple cases, no scale exists. Furthermore, it may happen that the scales which are internal means have minimum likelihood and those that are external means have maximum likelihood. Among Pearson Types are found both anomalous functions and functions which would be regarded as regular as regards maximum likelihood.

In this problem of scaling, likelihood is proportional to *a posteriori* probability with the *a priori* probability taken as constant.

MOMENTS OF ANY RATIONAL INTEGRAL ISOBARIC SAMPLE MOMENT FUNCTION

By Paul S. Dwyer

Introduction

The problem of moments of moments has been investigated by a number of authors. The assumption of an infinite universe (or that of a finite universe with replacements) permits the application of the "algebraic" method, the method of semi-invariants as introduced by Thiele (1) and developed by C. C. Craig (2) and the combinatorial analysis method introduced by R. A. Fisher (3) and used by N. St. Georgescu (4). A combinatorial analysis method has the particular advantage that it enables one to compute separate terms of a given formula.

The formulae for moments of moments have been simplified through the use of new moment functions. Thiele introduced the half-invariant (1) which resulted in considerable condensation. More recently Prof. R. A. Fisher (3) has introduced the sample function k whose expected value is a half invariant. The most compact formulization presented thus far is his formulation of the half invariants of the sample k_r in terms of the half invariants of the universe. This very compactness, however, makes it difficult to compare results with those expressed in the more conventional sample functions. Dr. Wishart has written a paper (7) in which he shows, among other things, how the Fisher results can be translated to the more conventional (Craig) results and vice versa, but such translation is in general no simple matter. It appears that the Fisher results are not immediately useful to the statistician who desires the formulae to be expressed in terms of the usual sample moment function. On the other hand the Fisher formulization is a remarkable discovery toward that harmony which must be naturally inherent in the field of moments of moments. Soper (6, 111) expressed the general situation when he wrote, "If the terrifying overgrowth of algebraic formulation accompanying this branch of statistical inquiry is destined to have a chief utility in induction and going back to causes, then perhaps Dr. Fisher's way of estimating a sample will prove to be most fertile, but if it is to be applied to problems of deduction, say to problems of successive eventuation such as propagation, then Mr. Craig's plain moments seem to have a firmer hold on the exigencies of time."

It would appear then that the Fisher formulae and the Craig formulae are both needed. Georgescu (4) showed a partial connection between them in applying to the m functions a combinatory analysis somewhat similar to that applied by R. A. Fisher to the k function. It is the purpose of the present

paper to work out a combinatorial procedure for a more general sample function so that either the Fisher or Georgescu combinatorial results come out as special cases. In making such a generalization no limitation is placed on the sample function except that it be rational integral and that all terms are of the same weight. Thus the results are applicable to m_r, $m_r + k_r$, $m_r k_r$, etc. as well as to m_r and k_r although they are not applicable to $\sqrt{m_r}$ or $\frac{m_r}{k_r}$. In this way the important formulae for the moments of a new sample moment function will be available by simple substitution as soon as any such new function is defined by a rational integral isobaric expansion of power sums.

It is thus the purpose of this paper to determine the moments of a general moment function of the sample. This is done by keeping the multipliers of the various partitions of power sums indefinite until all manipulation is complete. It is then possible to assign the definite values of these multipliers which are associated with the desired sample function and to obtain the moment of the desired moment function in this way. Thus the Fisher result $\kappa(42)$, and the Craig result $S_{11}(\nu_4, \nu_2)$ are special cases of the new result $\lambda_{11}(f_4, f_2)$. It is obvious that it is not possible to carry the results using these general moment functions as far as Fisher and Wishart (3), (5), (7), have carried the results of the decidedly advantageous (from the standpoint of simplicity of result) k function and yet it is surprising to find the simplicity which can be obtained in the general case. Incidentally the introduction of the more general symbols clarifies the successive steps of the partition analysis which are somewhat confusing in any specific case because of the insertion of the value of the coefficients of the power sums in which the sample moment function is expressed.

This paper is divided into three parts. The first part includes the necessary definitions, the basic formulae, and the general development of the algebraic method. In order to facilitate the algebraic work there is inserted a table giving the expected values of all possible partition products of power sums whose weight $\leqq 8$. The second part deals with the different sample functions which might be used. The third part gives a list of the various partition formulae, of weight $\leqq 8$, which contain no unit parts and shows how these can be used in writing the chief variations of the formulae for moments of moments.

Part I

1. **General Moment Functions.** Different moment functions have been defined in various ways, but all moment functions have in common the property that they may be expressed in terms of the power sums. It appears sensible to use this expression in terms of power sums as the working algebraic definition of moment functions. For example the function k_3, which is defined by R. A. Fisher to be that function of the sample whose expected value is the third cumulant (half invariant) is to be given the working definition of

$$k_3 = \frac{n(3)}{(n-1)(n-2)} - \frac{3(2)(1)}{(n-1)(n-2)} + \frac{2(1)(1)(1)}{n(n-1)(n-2)}$$

where the numerical expressions in parentheses indicate power sums of the sample.

Every term in the definition of a sample function has a "weight" which is equal to the sum of the power sums whose product is indicated by the term. Thus the weight of each of the terms of k_3 is 3. If all the terms of a given moment function have the same weight, the function is called isobaric and the weight of the function is equal to the weight of each term. Thus k_3 is an isobaric moment function and its weight is 3. Since all the functions so far proposed are isobaric we limit this generalization of moment functions to isobaric moment functions although it is possible that a more complex analysis could be worked out for non-isobaric functions.

Generality demands the inclusion of every possible partition product of power sums. Such generality can be obtained by writing

$$f_1 = a'_1(1)$$
$$f_2 = a'_2(2) + a'_{11}(1)^2$$
$$f_3 = a'_3(3) + a'_{21}(2)(1) + a'_{111}(1)^3$$
$$f_4 = a'_4(4) + a'_{31}(3)(1) + a'_{2^2}(2)^2 + a'_{21^2}(2)(1)^2 + a_{1^4}(1)^4$$

and in general

$$f_r = \sum a'_{p_1^{\pi_1} p_2^{\pi_2} p_s^{\pi_s}} (p_1)^{\pi_1} (p_2)^{\pi_2} \cdots (p_s)^{\pi_s}$$

where $(p_1)^{\pi_1} (p_2)^{\pi_2} \cdots (p_s)^{\pi_s}$ indicates any partition product of power sums, $a'_{p_1^{\pi_1} \cdots p_s^{\pi_s}}$ is its coefficient and the summation is taken for every possible partition. The number of parts of the partition is $\rho = \Sigma\pi$. It may be assumed, without loss of generality, that the partition is ordered, i.e.

$$p_1 \geqq p_2 \geqq p_3 \geqq \cdots \geqq p_s.$$

A natural numerical coefficient of each term is the number of ways the r units can be collected to form the given partition. This value is given by

$$\binom{1^r}{p_1^{\pi_1} p_2^{\pi_2} \cdots p_s^{\pi_s}} = \frac{r!}{(p_1!)^{\pi_1} (p_2!)^{\pi_2} \cdots (p_s!)^{\pi_s} \pi_1! \pi_2! \cdots \pi_s!}.$$

If we set

$$a'_{p_1^{\pi_1} \cdots p_s^{\pi_s}} = \binom{1^r}{p_1^{\pi_1} \cdots p_s^{\pi_s}} a_{p_1^{\pi_1} \cdots p_s^{\pi_s}}$$

the definition of f_r becomes

$$f_r = \sum \binom{1^r}{p_1^{\pi_1} \cdots p_s^{\pi_s}} a_{p_1^{\pi_1} \cdots p_s^{\pi_s}} (p_1)^{\pi_1} \cdots (p_s)^{\pi_s}$$

In the present paper the capital letters are used to represent the corresponding

functions of the universe *as defined by the corresponding power sums of the universe.* Thus

$$F_r = \sum \binom{1^r}{p_1^{\pi_1} \cdots p_s^{\pi_s}} a_{p_1^{\pi_1} \cdots p_s^{\pi_s}} (P_1)^{\pi_1} \cdots (P_s)^{\pi_s}$$

represents the corresponding function of the universe. In the case of the moment about the mean and the semi-invariant the Greek letters μ and λ have been used to represent the corresponding function of the universe. In the case of functions whose notation is quite widely established, it is preferable to use the conventional notation, but in introducing new functions it appears wise to use the relationship between small and capital letters since the correspondence between the English and Greek alphabets is not exactly one to one. It should be particularly noticed that this notation does not agree with a previously accepted scheme of using the small English letter to indicate the function whose expected value is indicated by the corresponding Greek letter. In the present paper it is not the expected value property which serves as the basis of notation but rather the definition of the function in terms of the partition products of power sums.

2. **The Working Definition of Moments About a Fixed Point.** The sample functions defined by

$$m_1' = \frac{(1)}{n}, \qquad m_2' = \frac{(2)}{n}, \qquad m_3' = \frac{(3)}{n}, \cdots, m_r' = \frac{(r)}{n}$$

are obtained from f_r by placing

$$a_{p_1^{\pi_1} \cdots p_s^{\pi_s}} = \begin{cases} \dfrac{1}{n} \text{ when } s = 1,\ \pi_1 = 1, \text{ and } p_1 = r. \\ 0 \text{ in all other cases.} \end{cases}$$

The Greek μ' is used to indicate the corresponding function of the universe.

3. **The Working Definition of Moments About the Mean.** The moments about the mean are defined by

$$m_1' = \frac{(1)}{n}, \qquad m_2 = \frac{(2)}{n} - \frac{(1)\,(1)}{n^2},$$

$$m_3 = \frac{(3)}{n} - \frac{3(2)\,(1)}{n^2} + \frac{2(1)^3}{n^3}, \qquad m_4 = \frac{(4)}{n} - \frac{4(3)\,(1)}{n^2} + \frac{6(2)\,(1)^2}{n^3} - \frac{3(1)^4}{n^4}$$

and in general m_r is obtained from f_r by placing

$$a_{p_1^{\pi_1} \cdots p_s^{\pi_s}} = \begin{cases} \dfrac{1}{n} \text{ if } s = 1,\ \pi_1 = 1, \text{ and } p_1 = r. \\ \dfrac{(-1)^{\pi_2}}{(n)^{1+\pi_2}} \text{ if } p_1 > 1,\ \pi_1 = 1,\ s = 2, \text{ and } p_2 = 1. \\ \dfrac{(-1)^{r-1}\,(r-1)}{n^r} \text{ if } p_1 = 1,\ s = 1, \text{ and } \pi_1 = r. \\ 0 \text{ in all other cases.} \end{cases}$$

The corresponding moments of the universe are indicated by the conventional μ. For conciseness moments about the mean are referred to as "moments."

4. **The Working Definition of the Half Invariants.** The half invariant moment functions of Thiele, as applied to the sample power sums are [see C. C. Craig (2, 7–10) and Frisch (12, 20–21)]

$$l_1' = \frac{(1)}{n}, \qquad l_2 = \frac{(2)}{n} - \frac{(1)\,(1)}{n^2}, \qquad l_3 = \frac{(3)}{n} - \frac{3(2)\,(1)}{n^2} + \frac{2(1)^3}{n^3}$$

$$l_4 = \frac{(4)}{n} - \frac{4(3)\,(1)}{n^2} - \frac{3(2)^2}{n^2} + \frac{12(2)\,(1)^2}{n^3} - \frac{6(1)^4}{n^4}$$

and in general

$$l_r = \sum \frac{(-1)^{\rho-1}\,(\rho-1)!}{n^\rho} \binom{1^r}{p_1^{\pi_1} \cdots p_s^{\pi_s}} (p_1)^{\pi_1} (p_2)^{\pi_2} \cdots (p_s)^{\pi_s}$$

so that

$$a_{p_1^{\pi_1} \cdots p_s^{\pi_s}} = \frac{(-1)^{\rho-1}\,(\rho-1)!}{n^\rho}.$$

The corresponding moments of the universe are indicated, after Thiele (1) and Craig (2), by λ. R. A. Fisher (3) used κ while Georgescu (4) used s.

In the present paper these functions are referred to as "Thiele moments."

5. **The k Functions of R. A. Fisher.** The k statistics of R. A. Fisher are defined in terms of the sample power sums by

$$k_1' = \frac{(1)}{n}, \qquad k_2 = \frac{(2)}{n-1} - \frac{(1)^2}{n(n-1)},$$

$$k_3 = \frac{n(3)}{(n-1)\,(n-2)} - \frac{3(2)\,(1)}{(n-1)\,(n-2)} + \frac{2(1)^3}{n^{(3)}}$$

$$k_4 = \frac{n(n+1)\,(4)}{(n-1)^{(3)}} - \frac{4(n+1)\,(3)\,(1)}{(n-1)^{(3)}} - \frac{3(2)^2}{(n-2)^{(2)}} + \frac{12(2)\,(1)^2}{(n-1)^{(3)}} - \frac{6(1)^4}{n^{(4)}}.$$

These values and values for k_5 and k_6 are given by R. A. Fisher (3, 203–4) while algebraic methods of attaining them are presented in sections 16, 17. They are referred to as Fisher moments. The corresponding functions of the universe, if used, would be represented by K_r.

6. **The h Function.** Just as Fisher introduced a sample function whose expected value is a Thiele moment of the universe, so it is possible to introduce a function whose expected value is a moment of the universe. Such a function is defined by

$$h_1' = \frac{(1)}{n}, \qquad h_2 = \frac{(2)}{n-1} - \frac{(1)^2}{n(n-1)},$$

$$h_3 = \frac{n(3)}{(n-1)(n-2)} - \frac{3(2)(1)}{(n-1)(n-2)} + \frac{2(1)^3}{n^{(3)}}$$

$$h_4 = \frac{(n^2-2n+3)(4)}{(n-1)^{(3)}} - \frac{4(n^2-2n+3)(3)(1)}{n^{(4)}} - \frac{3(2n-3)(2)^2}{n^{(4)}} + \frac{6(2)(1)^2}{(n-1)^{(3)}} - \frac{3(1)^4}{n^{(4)}}.$$

Methods of obtaining the expansion of this function in terms of power sums are presented in section 18. The corresponding function of the universe, if it were used, would be represented by H_r.

7. **Other Moment Functions.** It is possible to obtain an indefinite number of moment functions. For example one might define a function of weight 2 whose variance equals μ_4, (or μ_2^2). It is possible by the methods of this paper to find expressions for such moments.

For reference purposes Table I is provided showing the values of a for each partition of weight <6 for the functions m', m, l, h, k. The values of

$$\begin{pmatrix} 1^r \\ p_1^{\pi_1}\, p_2^{\pi_2} \cdots p_s^{\pi_s} \end{pmatrix}$$

are also inserted, in the left hand column, so that it is possible to read from the table the values for $f = m_r'$, m_r, l_r, k_r when $r < 6$.

8. **Products of f Functions.** The product of two or more isobaric functions is also isobaric and of weight equal to the sum of the weights of the functions. Thus

$$f_2 f_1 = [a_2(2) + a_{11}(1)(1)][a_1(1)] = a_2 a_1(2)(1) + a_{11} a_1(1)^2$$

$$f_2 f_1^2 = a_2 a_1^2(2)(1)^2 + a_{11} a_1^2(1)^4.$$

In multiplying f_{r_1} by f_{r_2} any term of f_{r_1} is of weight r_1 and when it is multiplied by any term of weight r_2, the result is a term of weight $r_1 + r_2$.

TABLE I

Coefficients of Products of Power Sums in the Expansion of Different Moment Functions

Numerical coefficient	a	m_r'	m_r	l_r	k_r	h_r
1	a_1	$\frac{1}{n}$	$\frac{1}{n}$	$\frac{1}{n}$	$\frac{1}{n}$	$\frac{1}{n}$
1	a_2	$\frac{1}{n}$	$\frac{1}{n}$	$\frac{1}{n}$	$\frac{1}{n-1}$	$\frac{1}{n-1}$
1	a_{11}	0	$\frac{-1}{n^2}$	$\frac{-1}{n^2}$	$\frac{-1}{n^{(2)}}$	$\frac{-1}{n^{(2)}}$
1	a_3	$\frac{1}{n}$	$\frac{1}{n}$	$\frac{1}{n}$	$\frac{n}{(n-1)^{(2)}}$	$\frac{n}{(n-1)^{(2)}}$
3	a_{21}	0	$\frac{-1}{n^2}$	$\frac{-1}{n^2}$	$\frac{-1}{(n-1)^{(2)}}$	$\frac{-1}{(n-1)^{(2)}}$
1	a_{111}	0	$\frac{2}{n^3}$	$\frac{2}{n^3}$	$\frac{2}{n^{(3)}}$	$\frac{2}{n^{(3)}}$
1	a_4	$\frac{1}{n}$	$\frac{1}{n}$	$\frac{1}{n}$	$\frac{n(n+1)}{(n-1)^{(3)}}$	$\frac{n^2-2n+3}{(n-1)^{(3)}}$
4	a_{31}	0	$\frac{-1}{n^2}$	$\frac{-1}{n^2}$	$-\frac{(n+1)}{(n-1)^{(3)}}$	$-\frac{n^2-2n+3}{n^{(4)}}$
3	a_{22}	0	0	$\frac{-1}{n^2}$	$\frac{-1}{(n-2)^{(2)}}$	$-\frac{2n-3}{n^{(4)}}$
6	a_{211}	0	$\frac{1}{n^3}$	$\frac{2}{n^3}$	$\frac{2}{(n-1)^{(3)}}$	$\frac{1}{(n-1)^{(3)}}$
1	a_{1111}	0	$\frac{-3}{n^4}$	$\frac{-6}{n^4}$	$\frac{-6}{n^{(4)}}$	$\frac{-3}{n^{(4)}}$
1	a_5	$\frac{1}{n}$	$\frac{1}{n}$	$\frac{1}{n}$	$\frac{n^2(n+5)}{(n-1)^{(4)}}$	$\frac{n(n^2-5n+10)}{(n-1)^{(4)}}$
5	a_{41}	0	$\frac{-1}{n^2}$	$\frac{-1}{n^2}$	$-\frac{n(n+5)}{(n-1)^{(4)}}$	$-\frac{n^2-5n+10}{(n-1)^{(4)}}$
10	a_{32}	0	0	$\frac{-1}{n^2}$	$-\frac{n(n-1)}{(n-1)^{(4)}}$	$-\frac{n-2}{(n-1)^{(4)}}$

TABLE I—*Concluded*

Numerical coefficient	a	m'_r	m_r	l_r	k_r	h_r
10	a_{311}	0	$\frac{1}{n^3}$	$\frac{2}{n^3}$	$\frac{2(n+2)}{(n-1)^{(4)}}$	$\frac{n^2-4n+8}{n^{(5)}}$
15	a_{221}	0	0	$\frac{2}{n^3}$	$\frac{2(n-1)}{(n-1)^{(4)}}$	$+\frac{(2n-4)}{n^{(5)}}$
10	a_{2111}	0	$\frac{-1}{n^4}$	$\frac{-6}{n^4}$	$-\frac{6}{(n-1)^{(4)}}$	$-\frac{1}{(n-1)^{(4)}}$
1	a_{11111}	0	$\frac{4}{n^5}$	$\frac{24}{n^5}$	$\frac{24}{n^{(5)}}$	$\frac{4}{n^{(5)}}$

R. A. Fisher [3, 207] used the product $k_3^2k_2$ as an illustration of the algebraic method. The more general $f_3^2f_2$ gives

$$\begin{aligned} f_3^2f_2 &= [a_3(3) + 3a_{21}(2)(1) + a_{111}(1)^3]^2[a_2(2) + a_{11}(1)(1)] \\ &= a_3^2a_2(3)(3)(2) + a_3^2a_{11}(3)(3)(1)(1) + 6a_3a_{21}a_2(3)(2)^2(1) \\ &+ [6a_3a_{21}a_{11} + 2a_3a_2a_{111}](3)(2)(1)^3 + 9a_{21}^2a_2(2)^3(1)^2 + 2a_3a_{111}a_{11}(3)(1)^5 \\ &+ [6a_{21}a_{111}a_2 + 9a_{21}^2a_{11}](2)^2(1)^4 + [6a_{21}a_{111}a_{11} + a_2a_{111}^3](2)(1)^6 + a_{111}^3a_{11}(1)^8 \end{aligned}$$

which reduces to the value as given by him when the values of a are substituted from Table I.

9. **The Expected Value of Any Partition Product.** The expected values of partition products are well known and are indicated by

$$\begin{aligned} E(p_1) &= n\mu'_{p_1} \\ E(p_1)(p_2) &= n\mu'_{p_1+p_2} + n(n-1)\mu'_{p_1}\mu'_{p_2} \\ E(p_1)(p_2)(p_3) &= n\mu'_{p_1+p_2+p_3} + n(n-1)\,[\mu'_{p_1+p_2}\mu'_{p_3} + \mu'_{p_1+p_3}\mu'_{p_2} + \mu'_{p_2+p_3}\mu'_{p_1}] \\ &\quad + n(n-1)\,(n-2)\,\mu'_{p_1}\mu'_{p_2}\mu'_{p_3}. \end{aligned}$$

and in general

$$E(p_1)^{\pi_1}(p_2)^{\pi_2}\cdots(p_s)^{\pi_s} = \sum n^{(\tau)} \begin{pmatrix} p_1^{\pi_1}\, p_2^{\pi_2} \cdots p_s^{\pi_s} \\ q_1^{\chi_1}\, q_2^{\chi_2} \cdots q_t^{\chi_t} \end{pmatrix} (\mu'_{q_1})^{\chi_1}(\mu'_{q_2})^{\chi_2}\cdots(\mu'_{q_t})^{\chi_t}$$

where $\tau = \chi_1 + \chi_2 + \chi_3 + \cdots + \chi_t$ and $\begin{pmatrix} p_1^{\pi_1}\, p_2^{\pi_2} \cdots p_s^{\pi_s} \\ q_1^{\chi_1}\, q_2^{\chi_2} \cdots q_t^{\chi_t} \end{pmatrix}$ indicates the

number of ways in which the partition $p_1^{\pi_1} p_2^{\pi_2} \cdots p_s^{\pi_s}$ can be grouped to form the partition $q_1^{\chi_1} q_2^{\chi_2} \cdots q_t^{\chi_t}$.

The continued application of the result above leads to a large number of formulae. In order to make these results accessible I present in Table II the expected values of all partition products of weight $\leqq 8$. The essence of the table is the evaluation of the expression $\begin{pmatrix} p_1^{\pi_1} p_2^{\pi_2} \cdots p_s^{\pi_s} \\ q_1^{\chi_1} q_2^{\chi_2} \cdots q_t^{\chi_t} \end{pmatrix}$. The numbers at the top of each column indicate the subscripts of the μ's which must, of course, be multiplied by $n^{(\tau)}$. The entries on the extreme left are the numerical coefficients associated with each row.

10. **The Expected Values of the f Functions.** With the use of Table II one is able to write expressions for the expected values of f_r when $r < 9$.

$$\mu_1'(f_1) = E(f_1) = a_1 n\mu_1'$$

$$\mu_1'(f_2) = E(f_2) = (a_2 + a_{11})n\mu_2' + a_{11}n(n-1)\mu_1'^2$$

$$\mu_1'(f_3) = E(f_3) = (a_3 + 3a_{21} + a_{111})n\mu_3' + 3(a_{21} + a_{111})n(n-1)\mu_2'\mu_1' + a_{111}n(n-1)(n-2)\mu_1'^3 \text{ etc.}$$

If the expected values of the f functions are expressed in terms of the moments about the mean of the universe, these formulae become, since $\mu_1' = 0$

$$\mu_1'(f_1) = 0$$

$$\mu_1'(f_2) = (a_2 + a_{11})n\mu_2$$

$$\mu_1'(f_3) = (a_3 + 3a_{21} + a_{111})n\mu_3$$

$$\mu_1'(f_4) = (a_4 + 4a_{31} + 3a_{22} + 6a_{211} + a_{1111})n\mu_4 + 3(a_{22} + 2a_{211} + a_{1111})n(n-1)\mu_2^2 \text{ etc.}$$

These may be written more symbolically as

$$\mu_1'(f_1) = 0$$

$$\mu_1'(f_2) = b_2 n\mu_2$$

$$\mu_1'(f_3) = b_3 n\mu_3$$

$$\mu_1'(f_4) = b_4 n\mu_4 + 3b_{22}n(n-1)\mu_2^2 \text{ etc.}$$

11. **The Expected Value of Products of f Functions.** The expected value of products of f functions may be similarly found. For example

$$\mu_2'(f_2) = E(f_2^2) = E[a_2(2) + a_{11}(1)^2]^2 = a_2^2 E(2)^2 + 2a_2 a_{11} E(2)(1)(1) + a_{11}^2 E(1)^4.$$

TABLE II

Expected Values of Partition Products

weight = 1

coef.		n
	π	1
1	1	1

weight = 2

coef.		n	$n^{(2)}$
	π	2	11
1	2	1	
1	11	1	1

weight = 3

coef.		n	$n^{(2)}$	$n^{(3)}$
	π	3	21	111
1	3	1		
3	21	1	1	
1	1^3	1	3	1

weight = 4

coef.		n	$n^{(2)}$	$n^{(2)}$	$n^{(3)}$	$n^{(4)}$
	π	4	31	22	211	1^4
1	4	1				
4	31	1	1			
3	22	1		1		
6	211	1	2	1	1	
1	1111	1	4	3	6	1

weight = 5

coef.		n	$n^{(2)}$	$n^{(2)}$	$n^{(3)}$	$n^{(3)}$	$n^{(4)}$	$n^{(5)}$
	π	5	41	32	31^2	$2^2\,1$	21^3	1^5
1	5	1						
5	41	1	1					
10	32	1		1				
10	311	1	2	1	1			
15	221	1	1	2		1		
10	2111	1	3	4	3	3	1	
1	1^5	1	5	10	10	15	10	1

TABLE II—*Continued*

weight = 6

coef.		n	$n^{(2)}$	$n^{(2)}$	$n^{(2)}$	$n^{(3)}$	$n^{(3)}$	$n^{(3)}$	$n^{(4)}$	$n^{(4)}$	$n^{(5)}$	$n^{(6)}$
	π	6	51	42	33	411	321	2^3	31^3	$2^2\,1^2$	21^4	1^6
1	6	1										
6	51	1	1									
15	42	1		1								
10	33	1			1							
15	411	1	2	1		1						
60	321	1	1	1	1		1					
15	2^3	1		3				1				
20	31^3	1	3	3	1	3	3		1			
45	$2^2\,1^2$	1	2	3	2	1	4	1		1		
15	21^4	1	4	7	4	6	16	3	4	6	1	
1	1^6	1	6	15	10	15	60	15	20	45	15	1

weight = 7

coef.		n	$n^{(2)}$	$n^{(2)}$	$n^{(2)}$	$n^{(3)}$	$n^{(3)}$	$n^{(3)}$	$n^{(3)}$	$n^{(4)}$	$n^{(4)}$	$n^{(4)}$	$n^{(5)}$	$n^{(5)}$	$n^{(6)}$	$n^{(7)}$
	π	7	61	52	43	51^2	421	$3^2\,1$	32^2	41^3	321^2	$2^3\,1$	31^4	$2^2\,1^3$	21^5	1^7
1	7	1														
7	61	1	1													
21	52	1		1												
35	43	1			1											
21	51^2	1	2	1		1										
105	421	1	1	1	1		1									
70	$3^2\,1$	1	1		2			1								
105	32^2	1		2	1				1							
35	41^3	1	3	3	1	3	3			1						
210	321^2	1	2	2	3	1	2	2	1		1					
105	$2^3\,1$	1	1	3	3		3		3			1				
35	31^4	1	4	6	5	6	12	4	3	4	6		1			
105	$2^2\,1^3$	1	3	5	7	3	9	6	7	1	6	3		1		
21	21^5	1	5	11	15	10	35	20	25	10	40	15	5	10	1	
1	1^7	1	7	21	35	21	105	70	105	35	210	105	35	105	21	1

TABLE II—*Concluded*

weight = 8

$f_3^2 f_2$	coef.		n	$n^{(2)}$	$n^{(2)}$	$n^{(2)}$	$n^{(2)}$	$n^{(3)}$	$n^{(3)}$	$n^{(3)}$	$n^{(3)}$	$n^{(3)}$	$n^{(4)}$	$n^{(4)}$	$n^{(4)}$	$n^{(4)}$	$n^{(4)}$	$n^{(5)}$	$n^{(5)}$	$n^{(5)}$	$n^{(6)}$	$n^{(6)}$	$n^{(7)}$	$n^{(8)}$
		π	8	71	62	53	44	611	521	431	42^2	$3^2\,2$	51^3	421^2	$3^2\,1^2$	$32^2\,1$	2^4	41^4	321^3	$2^3\,1^2$	31^5	$2^2\,1^4$	21^6	1^8
	1	8	1																					
	8	71	1	1																				
	28	62	1		1																			
	56	53	1			1																		
	35	44	1				1																	
	28	61^2	1	2	1			1																
	168	521	1	1	1	1			1															
	280	431	1	1		1	1			1														
	210	42^2	1		2		1				1													
$a_3^2 a_2$	280	$3^2\,2$	1		1	2						1												
	56	51^3	1	3	3	1		3	3				1											
	420	421^2	1	2	2	2	1	1	2	2	1			1										
$a_3^2 a_{11}$	280	$3^2\,1^2$	1	2	1	2	2	1		4		1			1									
$6a_3a_{21}a_2$	840	$32^2\,1$	1	1	2	3	1		2	1	1	2				1								
	105	2^4	1		4		3				6						1							
	70	41^4	1	4	6	4	1	6	12	4	3		4	6				1						
$6a_3a_{21}a_{11} + 2a_3a_{111}a_2$	560	321^3	1	3	4	5	3	3	6	9	3	4	1	3	3	3			1					
$9a_{21}^2 a_2$	420	$2^3\,1^2$	1	2	4	6	3	1	6	6	6	6		3		6	1			1				
$2a_3a_{111}a_{11}$	56	31^5	1	5	10	11	5	10	30	25	15	10	10	30	10	15		5	10		1			
$6a_{21}a_{111}a_2 + 9a_{21}^2 a_{11}$	210	$2^2\,1^4$	1	4	8	12	7	6	20	28	16	20	4	18	12	28	3	1	8	6		1		
$6a_{21}a_{111}a_{11} + a_{111}^2 a_2$	28	21^6	1	6	16	26	15	15	66	90	60	70	20	105	60	150	15	15	80	45	6	15	1	
$a_{111}^2 a_{11}$	1	1^8	1	8	28	56	35	28	168	280	210	280	56	420	280	840	105	70	560	420	56	210	28	1

Table II can now be used by indicating a_2^2 as a multiplier of $E(2)^2$, $2a_2a_{11}$ as a multiplier of $E(2)(1)(1)$ and a_{11}^2 as a multiplier of $(1)^4$. Then at once it is evident that

$$\begin{aligned}\mu_2'(f_2) &= (a_2^2 + 2a_2a_{11} + a_{11}^2)n\mu_4 + (a_2^2 + 2a_2a_{11} + 3a_{11}^2)n(n-1)\mu_2^2 \\ &= (a_2 + a_{11})^2 n\mu_4 + [(a_2 + a_{11})^2 + 2a_{11}^2]n(n-1)\mu_2^2 \\ &= b_2^2 n\mu_4 + (b_2^2 + 2b_{11}^2)n(n-1)\mu_2^2.\end{aligned}$$

Similarly

$$\mu_{11}'(f_3, f_2) = b_3b_2n\mu_5 + (b_3b_2 + 3b_{21}b_2 + 6b_{21}b_{11})n(n-1)\mu_3\mu_2$$

$$\begin{aligned}\mu_2'(f_3) = b_3^2n\mu_6 &+ (9b_{21}^2 + 6b_3b_{21})n(n-1)\mu_4\mu_2 + (b_3^2 + 9b_{21}^2)n(n-1)\mu_3^2 \\ &+ (9b_{21}^2 + 6b_{111}^2)n(n-1)(n-2)\mu_2^3\end{aligned}$$

etc.

where $b_3 = a_3 + 3a_{21} + a_{111}$, $b_{21} = a_{21} + a_{111}$, $b_{111} = a_{111}$. The important special cases are obtained by assigning the proper values to the a's as given in Table I. Thus

$$\mu_2'(m_2) = \frac{1}{n^3}[(n-1)^2\mu_4 + (n^2 - 2n + 3)(n-1)\mu_2^2]$$

which agrees with the corrected result of "Student" in 1908 (8, 3) and Tchouproff (10, 192). Similarly

$$\mu_{11}'(m_3, m_2) = \frac{1}{n^4}[(n-1)^2(n-2)\mu_5 + (n-1)(n-2)(n^2 - 5n + 10)\mu_3\mu_2]$$

$$\begin{aligned}\mu_2'(m_3) = \frac{1}{n^5}[&(n-1)^2(n-2)^2\mu_6 + (-6n+15)(n-1)(n-2)^2\mu_4\mu_2 \\ &+ (n^2 - 2n + 10)(n-1)(n-2)^2\mu_3^2 + (9n^2 - 36n + 60)(n-1)(n-2)\mu_2^3]\end{aligned}$$

etc.

In the same way

$$\mu_2'(k_2) = \frac{\mu_4}{n} + \frac{(n^2 - 2n + 3)\mu_2^2}{n(n-1)}$$

$$\mu_{11}'(k_3, k_2) = \frac{\mu_5}{n} + \frac{(n^2 - 5n + 10)\mu_3\mu_2}{n(n-1)}$$

$$\mu_2'(k_3) = \frac{\mu_6}{n} + \frac{(-6n+15)\mu_4\mu_2}{n(n-1)} + \frac{(n^2 - 2n + 10)\mu_3^2}{n(n-1)} + \frac{(9n^2 - 36n + 60)\mu_2^3}{n(n-1)(n-2)}$$

etc.

and

$$\mu_2'(m_2') = \frac{1}{n}[\mu_4 + (n-1)\mu_2^2]$$

$$\mu_{11}'(m_3', m_2') = \frac{1}{n}[\mu_5 + (n-1)\mu_3\mu_2]$$

$$\mu_2'(m_3') = \frac{1}{n}[\mu_6 + (n-1)\mu_3^2]$$

etc.

12. **The Expected Value of the Products of f Functions in Terms of the Thiele Moments of the Universe.** The formulae giving the μ's in term if the λ's are

$$\mu_2 = \lambda_2$$

$$\mu_3 = \lambda_3$$

$$\mu_4 = \lambda_4 + 3\lambda_2^2$$

$$\mu_5 = \lambda_5 + 10\lambda_3\lambda_2$$

$$\mu_6 = \lambda_6 + 15\lambda_4\lambda_2 + 10\lambda_3^2 + 15\lambda_2^3$$

.....

$$\mu_r = \sum \binom{1^r}{p_1^{\pi_1} p_2^{\pi_2} \cdots p_s^{\pi_s}} (\lambda_{p_1})^{\pi_1} (\lambda_{p_2})^{\pi_2} \cdots (\lambda_{p_s})^{\pi_s}$$

where the summation holds for those partitions having no unit parts. See the results of Craig (2, 7–11) and Frisch (12, 21). It is at once possible to express the moment formulae in terms of the Thiele moments of the universe. Thus the general results above become

$$\mu_2'(f_2) = b_2^2 n\lambda_4 + [3b_2^2 n + (b_2^2 + 2b_{11}^2)n(n-1)]\lambda_2^2$$

$$\mu_{11}'(f_3, f_2) = b_3 b_2 n\lambda_5 + [10b_3 b_2 n + (b_3 b_2 + 3b_{21}b_2 + 6b_{21}b_{11})n(n-1)]\lambda_3\lambda_2$$

$$\mu_2'(f_3) = b_3^2 n\lambda_6 + [15b_3^2 n + (9b_{21}^2 + 6b_3 b_{21})n(n-1)]\lambda_4\lambda_2 + [10b_3^2 n + (b_3^2 + 9b_{21}^2)n(n-1)]\lambda_3^2 + [15b_3^2 n + (27b_{21}^2 + 18b_3 b_{21})n(n-1) + (9b_{21}^2 + 6b_{111}^2)n(n-1)(n-2)]\lambda_2^3.$$

13. **The Thiele Moments of the f's in terms of Thiele Moments.** It is now possible to reduce to the Thiele moments of the f's by means of the usual relations

$$\lambda_2(f_r) = \mu_2(f_r) - \mu_1'^2(f_r)$$

$$\lambda_{11}(f_{r_1}, f_{r_2}) = \mu_{11}'(f_{r_1}, f_{r_2}) - \mu_{10}'(f_{r_1}, f_{r_2})\mu_{01}'(f_{r_1}, f_{r_2})$$

$$\lambda_3(f_r) = \mu_3'(f_r) - 3\mu_2'(f_r)\mu_1'(f_r) + 2\mu_1'^3(f_r)$$

etc.

so that the results become

$$\lambda_2(f_2) = b_2^2 n\lambda_4 + 2[b_2^2 n + b_{11}^2 n(n-1)]\lambda_2^2$$

$$\lambda_{11}(f_3, f_2) = b_3 b_2 n\lambda_5 + \{3[b_3 b_2 n + b_{21} b_2 n(n-1)] + 6[b_3 b_2 n + b_{21} b_{11} n(n-1)]\}\lambda_3\lambda_2$$

$$\lambda_2(f_3) = b_3^2 n\lambda_6 + \{6[b_3^2 n + b_3 b_{21} n(n-1)] + 9[b_3^2 n + b_{21}^2 n(n-1)]\}\lambda_4\lambda_2$$
$$+ 9[b_3^2 n + b_{21}^2 n(n-1)]\lambda_3^2 + \{9[b_3^2 n + 2b_3 b_{21} n(n-1) + b_{21}^2 n(n-1) + b_{21}^2 n^{(3)}]$$
$$+ 6[b_3^2 n + 3b_{21}^2 n(n-1) + b_{111}^2 n(n-1)(n-2)]\}\lambda_2^3$$

etc.

The formulae as written are adapted to the partition representation of Part III.

When the f's are equal to the m's we have

$$\lambda_2(m_2) = \frac{(n-1)^2\lambda_4}{n^3} + \frac{2(n-1)\lambda_2^2}{n^2}.$$

$$\lambda_{11}(m_3, m_2) = \frac{(n-1)^2(n-2)\lambda_5}{n^4} + \frac{6(n-1)(n-2)\lambda_3\lambda_2}{n^3}$$

$$\lambda_2(m_3) = \frac{(n-1)^2(n-2)^2\lambda_6}{n^5} + \frac{9(n-1)(n-2)^2\lambda_4\lambda_2}{n^4}$$
$$+ \frac{9(n-1)(n-2)^2\lambda_3^2}{n^4} + \frac{6(n-1)(n-2)\lambda_2^3}{n^3}$$

etc.

which are the results as previously given by C. C. Craig (2, 55). In like manner when the $f_r = k_r$

$$\lambda_2(k_2) = \frac{\lambda_4}{n} + \frac{2\lambda_2^2}{n-1}$$

$$\lambda_{11}(k_3, k_2) = \frac{\lambda_5}{n} + \frac{6\lambda_3\lambda_2}{n-1}$$

$$\lambda_2(k_3) = \frac{\lambda_6}{n} + \frac{9\lambda_4\lambda_2}{n-1} + \frac{9\lambda_3^2}{n-1} + \frac{6n\lambda_2^3}{(n-1)(n-2)}$$

etc.

as given by R. A. Fisher [3, 210] while

$$\lambda_2(m_2') = \frac{1}{n}(\lambda_4 + 2\lambda_2^2)$$

$$\lambda_{11}(m_3', m_2') = \frac{1}{n}(\lambda_5 + 9\lambda_3\lambda_2)$$

$$\lambda_2(m_3') = \frac{1}{n}(\lambda_6 + 15\lambda_4\lambda_2 + 9\lambda_3^2 + 15\lambda_2^3).$$

etc.

14. **Various Formulization of Results.** Although different moment functions of the universe may be used it is customary to express the results in terms of universe moments about a fixed point, in terms of universe moments, or in terms of universe Thiele moments. It is possible to express results in any of the nine forms

$$\left.\begin{matrix}\mu'(f_r)\\ \mu(f_r)\\ \lambda(f_r)\end{matrix}\right\} \text{ in terms of } \left\{\begin{matrix}\text{moments about a fixed point } (\mu')\\ \text{moments } (\mu)\\ \text{Thiele moments } (\lambda)\end{matrix}\right.$$

where f_r represents the isobaric sample moment function of weight r. One purpose of such varied formulization is to discover the most compact form and also the one best adapted to use in the case of a normal universe or a universe whose moments obey some discoverable law. As suggested above Craig (2) has shown the relative compactness obtained by using $\lambda(m_r)$ and Thiele moments of the universe while R. A. Fisher (3) has shown the great additional compactness obtained by taking $f_r = k_r$.

15. **The Application of the Algebraic Method to $\lambda_{21}(f_3, f_2)$.** Before leaving the algebraic method it is perhaps wise to outline the steps in the case of a more involved problem. We take the example which R. A. Fisher (3, 207) has used in the case in which $f_r = k_r$. To find $\lambda_{21}(f_3, f_2)$.

The value of $f_3^2 f_2$ was found in section 8. To find its expected value it is only necessary to enter the coefficients of the different partition products in this expansion at the left of the corresponding rows as indicated in Table II.

The coefficient of any moment partition of the universe is found by multiplying each column entry by its corresponding left row entry and then by multiplying by $n^{(r)}$ as indicated at the top. Thus the coefficient of μ_8' is

$$(a_3^2a_2 + a_3^2a_{11} + 6a_3a_{21}a_2 + 6a_3a_{21}a_{11} + 2a_3a_{111}a_2 + 9a_{21}^2a_2 + 2a_3a_{111}a_{11} + 6a_2a_{21}a_{111} + 9a_{21}a_{21}a_{11} + 6a_{21}a_{111}a_{11} + a_{111}^2a_2 + a_{111}^2a_{11})n$$

which after some algebraic work reduces to

$$(a_3 + 3a_{21} + a_{111})^2(a_2 + a_{11})n = b_3^2b_2n.$$

In this manner it is possible to write the result either in terms of universe moments about a fixed point or in terms of universe moments. If moments are used, one may neglect all column partitions involving unity.

It should be noted that the a's defining k_r as given in Table I can be inserted here if desired. If these multipliers are introduced throughout the rows and columnar partitions involving unit parts are not used one will arrive at Table I of R. A. Fisher [3, 208] though there are some slight typographical errors in his rows for $(3)^2$ $(1)^2$ and (3) (2^2) (1).

Determining all the coefficients in this manner we find after considerable algebraic manipulation that

$$\mu'_{21}(f_3, f_2) = b_3^2 b_2 n\mu_8 + [b_3^2 b_2 + 9b_{21}^2 b_2 + 12b_2 b_{21} b_{11} + 6b_3 b_{21} b_2]n(n-1)\mu_6\mu_2$$
$$+ [2b_3^2 b_2 + 18b_{21}^2 b_2 + 18b_{21}^2 b_{11} + 6b_3 b_{21} b_2 + 12b_3 b_{21} b_{11}]n(n-1)\mu_5\mu_3$$
$$+ [2b_3^2 b_{11} + 9b_{21}^2 b_2 + 18b_{21}^2 b_{11} + 6b_3 b_{21} b_2]n(n-1)\mu_4^2 + [36b_{21}^2 b_2$$
$$+ 54b_{21}^2 b_{11} + 6b_3 b_{21} b_2 + 12b_3 b_{21} b_{11} + 12b_3 b_{111} b_{11} + 72b_{21} b_{111} b_{11}$$
$$+ 18b_{111}^2 b_2]n(n-1)(n-2)\mu_4\mu_2^2 + [b_3^2 b_2 + 6b_3 b_{21} b_2 + 12b_3 b_{21} b_{11}$$
$$+ 27b_{21}^2 b_2 + 90b_{21}^2 b_{11} + 36b_{21} b_{111} b_2 + 72b_{21} b_{111} b_{11} + 36b_{111}^2 b_{11}]n(n-1)(n-2)\mu_3^2\mu_2$$
$$+ [9b_{21}^2 b_2 + 18b_{21}^2 b_{11} + 36b_{21} b_{111} b_{11} + 6b_{111}^2 b_2 + 36b_{111}^2 b_{11}]n(n-1)(n-2)(n-3)\mu_2^4.$$

If $f_r = k_r$ the proper values of b are inserted and the expression above becomes that given by R. A. Fisher (3, 208). For example the coefficient of μ_2^4 is

$$\frac{(9n^3 - 63n^2 + 240n - 420)(n-3)}{n^2(n-1)^2(n-2)}$$

when

$$b_2 = \frac{1}{n}, \qquad b_3 = \frac{1}{n}, \qquad b_{11} = -\frac{1}{n(n-1)}$$

$$b_{21} = -\frac{1}{n(n-1)}, \qquad b_{111} = \frac{2}{n(n-1)(n-2)}.$$

The algebraic results involved in changing the general formula above to other functions are too extended to present here. A symbolic means of attaining them is included in later sections of the paper.

Part II. The Determination of Specific f Functions

16. **Functions Determined by the b's.** In Part I it was shown how various f functions are defined by giving definite values to the coefficients of the power sums. It is the purpose of this part of the paper to show how functions can be specified by means of their expected values in terms of moments of the universe. This is essentially the method used by R. A. Fisher in defining his k function and it is here extended to other functions. In this case the b's are first determined and the a's are then found from them. The first moments of f_1, f_2, f_3 were given in section 10. To these we add, as shown by Table II

$$\mu'_1(f_4) = (a_4 + 4a_{31} + 3a_{22} + 6a_{211} + a_{1111})n\mu'_4 + 4(a_{31} + 3a_{211} + a_{1111})n(n-1)\mu'_3\mu'_1$$
$$+ 3(a_{22} + 2a_{211} + a_{1111})n(n-1)\mu_2'^2 + 6(a_{211} + a_{1111})n(n-1)(n-2)\mu'_2\mu_1'^2$$
$$+ a_{1111}n(n-1)(n-2)(n-3)\mu_1'^4$$

etc.

These can be written more symbolically in terms of the b's

$$\mu_1'(f_1) = b_1 n\mu_1'$$

$$\mu_1'(f_2) = b_2 n\mu_2' + b_{11}n(n-1)\mu_1'^2$$

$$\mu_1'(f_3) = b_3 n\mu_3' + 3b_{21}n(n-1)\mu_2'\mu_1' + b_{111}n(n-1)(n-2)\mu_1'^3$$

$$\mu_1'(f_4) = b_4 n\mu_4' + 4b_{31}n(n-1)\mu_3'\mu_1' + 3b_{22}n(n-1)\mu_2'^2 + 6b_{211}n^{(3)}\mu_2'\mu_1'^2 + 6n^{(4)}\mu_1'^4,$$

and in general

$$\mu_1'(f_r) = \sum \binom{1^r}{p_1^{\pi_1}\, p_2^{\pi_2} \cdots p_s^{\pi_s}} b_{p_1^{\pi_1} \cdots p_s^{\pi_s}} n^{(\rho)} (\mu_{p_1}')^{\pi_1} (\mu_{p_2}')^{\pi_2} \cdots (\mu_{p_s}')^{\pi_s}.$$

The expansion of the function in terms of the power sums of the sample demands the determination of the a's. This can be accomplished by solving the equations

$$\begin{aligned}
a_1 &= b_1 \\
a_2 + a_{11} &= b_2 \\
a_{11} &= b_{11} \\
a_3 + 3a_{21} + a_{111} &= b_3 \\
a_{21} + a_{111} &= b_{21} \\
a_{111} &= b_{111} \\
a_4 + 4a_{31} + 3a_{22} + 6a_{211} + a_{1111} &= b_4 \\
a_{31} + 3a_{211} + a_{1111} &= b_{31} \\
a_{22} + 2a_{211} + a_{1111} &= b_{22} \\
&\text{etc.}
\end{aligned}$$

The solutions are

$$\begin{aligned}
a_1 &= b_1 \\
a_2 &= b_2 - b_{11} \\
a_{11} &= b_{11} \\
a_3 &= b_3 - 3b_{21} + 2b_{111} \\
a_{21} &= b_{21} - b_{111} \\
a_{111} &= b_{111} \\
a_4 &= b_4 - 4b_{31} - 3b_{22} + 12b_{211} - 6b_{1111} \\
a_{31} &= b_{31} - 3b_{211} + 2b_{1111} \\
a_{22} &= b_{22} - 2b_{211} + b_{1111} \\
a_{211} &= b_{211} - b_{1111} \\
a_{1111} &= b_{1111}.
\end{aligned}$$

The values of a_r, at least for $r \leqq 4$, follow the law

$$a_r = \sum \binom{1^r}{p_1^{\pi_1} \cdots p_s^{\pi_s}} (-1)^{\rho-1} (\rho - 1)\, !\, b_{p_1^{\pi_1} \cdots p_s^{\pi_s}}$$

and

$a_{21} = \overline{|a_2 a_1|}$ where $\overline{|a_2 a_1|}$ indicates that $a_2 = b_2 - b_{11}$ is multiplied by $a_1 = b_1$, the rule of multiplication being suffixing of subscripts. Similarly $a_{22} = \overline{|a_2 a_2|} = \overline{|(b_2 - b_{11})\,(b_2 - b_{11})|} = b_{22} - 2b_{211} + b_{1111}$.

This statement illustrates a general theorem which will be established later in another paper by a different approach that for all cases

$$a_r = \sum \binom{1^r}{p_1^{\pi_1} \cdots p_s^{\pi_s}} (-1)^{\rho-1} (\rho - 1)\, !\, b_{p_1^{\pi_1} \cdots p_s^{\pi_s}}$$

and that

$$a_{r_1 \cdots r_t} = \overline{|a_{r_1}\, a_{r_2} \cdots a_{r_t}|}.$$

This theorem enables one to write, with comparative ease, the coefficient of any product of power sums in a sample function whose expected values is defined. For example the functional coefficient of (3)(2) in f_5 is

$$\overline{|a_3 a_2|} = \overline{|(b_3 - 3b_{21} + 2b_{111})\,(b_2 - b_{11})|} = b_{32} - b_{311} - 3b_{221} + 5b_{2111} - 2b_{11111}$$

while that of (3)(1)(1) is $\overline{|a_3 a_1 a_1|} = b_{311} - 3b_{2111} + 2b_{11111}$. If the expected value of the function is known the b's are determined and the values of the above expressions can be found by substitution.

17. **The Values of the Fisher Moments (k functions).** The k functions have been defined to be these functions whose expected values are the Thiele moments of the universe. Thus $\mu_1'(k_r) = \lambda_r$ and since

$$\lambda_r = \sum \binom{1^r}{p_1^{\pi_1} p_2^{\pi_2} \cdots p_s^{\pi_s}} (-1)^{\rho-1} (\rho - 1)\, !\, (\mu'_{p_1})^{\pi_1} (\mu'_{p_2})^{\pi_2} \cdots (\mu'_{p_s})^{\pi_s}.$$

it follows at once that by comparison with $\mu_1'(f_r)$ in the last section, that

$$b_{p_1^{\pi_1} p_2^{\pi_2} \cdots p_s^{\pi_s}} = \frac{(-1)^{\rho-1} (\rho - 1)\, !}{n^{(\rho)}}.$$

Thus

$$b_1 = \frac{1}{n}; \quad b_2 = \frac{1}{n}; \quad b_{11} = -\frac{1}{n^{(2)}}; \quad b_3 = \frac{1}{n}; \quad b_{21} = \frac{-1}{n^{(2)}}; \quad b_{111} = \frac{2}{n^{(3)}};$$

$$b_4 = \frac{1}{n}; \quad b_{31} = \frac{-1}{n^{(2)}}; \quad b_{22} = \frac{-1}{n^{(2)}}; \quad b_{211} = \frac{2}{n^{(3)}}; \quad b_{1111} = \frac{-6}{n^{(4)}}; \text{ etc.}$$

The insertion of these values in the formulae of section 16 gives the values of a such as those indicated in Table I and in section 5. Thus the coefficient of (3)(2) in f_5 is

$$10(b_{32} - b_{311} - 3b_{221} + 5b_{2111} - 2b_{11111}) = -10\left[\frac{1}{n^{(2)}} + \frac{2}{n^{(2)}} + \frac{6}{n^{(3)}} + \frac{30}{n^{(4)}} + \frac{48}{n^{(5)}}\right]$$

$$= -\frac{10n^{(2)}}{(n-1)^{(4)}}.$$

The coefficient of (3)(1)(1) is

$$10(b_{311} - 3b_{2111} + 2b_{11111}) = 10\left[\frac{2}{n^{(3)}} + \frac{18}{n^{(4)}} + \frac{48}{n^{(5)}}\right] = \frac{10(2n+4)}{(n-1)^{(4)}}.$$

18. **The h Functions.** It is also possible to define a function whose expected value is the moment of the universe. Thus $\mu_1'(h_r) = \mu_r$ where

$$\mu_r = \sum \binom{1^r}{p_1^{\pi_1} \cdots p_s^{\pi_s}} A_{p_1^{\pi_1} \cdots p_s^{\pi_s}} (\mu'_{p_1})^{\pi_1} (\mu'_{p_2})^{\pi_2} \cdots (\mu'_{p_s})^{\pi_s}$$

and

$$A_{p_1^{\pi_1} \cdots p_s^{\pi_s}} = \begin{cases} 1 \text{ if } s = 1, \pi_1 = 1, \text{ and } p_1 = r. \\ (-1)^{\pi_2} \text{ if } p_1 > 1, \pi_1 = 1, s = 2 \text{ and } p_2 = 1. \\ (-1)^{r-1}(r-1) \text{ if } p_1 = 1, s = 1, \text{ and } \pi_1 = r. \\ 0 \text{ in all other cases.} \end{cases}$$

Comparing with the value of $\mu_1'(f_r)$ in section 16 we have

$$b_{b_1^{\pi_1} \cdots b_s^{\pi_s}} = \frac{A_{p_1^{\pi_1} \cdots p_s^{\pi_s}}}{n^{(\rho)}}.$$

The substitution of these values of b in the results of section 16 gives the expansions of h_r in terms of power sums as illustrated by the formulae of section 6 and Table I. Thus the coefficient of (3)(2) is

$$10(b_{32} - b_{311} - 3b_{221} + 5b_{2111} - 2b_{11111})$$

$$= -10\left[0 + \frac{1}{n^{(3)}} + 0 + \frac{5}{n^{(4)}} + \frac{8}{n^{(5)}}\right] = \frac{-10(n-2)}{(n-1)^{(4)}}.$$

Similarly the coefficient of (3)(1)(1) in h_5 is

$$10(b_{311} - 3b_{2111} + 2b_{11111}) = 10\left[\frac{1}{n^{(3)}} + \frac{3}{n^{(4)}} + \frac{8}{n^{(5)}}\right] = \frac{10(n^2 - 4n + 8)}{n^{(5)}}.$$

19. **The h' Functions.** One line of attack calls for the introduction of new moment functions which will result in simpler formulae. Thus for example,

C. C. Craig wrote (2, 37) "It rather seems that the best hopes of effectively further simplifying the problem of sampling for statistical characteristics lie either in the discovery of a new kind of symmetric function of all the observations which may be used to characterize frequency functions and which will be more amenable than either moments or semi-invariants for use in sampling problems, or in, what may very well prove to be much better and more feasible, the abandonment of the method of characterizing frequency functions by symmetric functions of all the observations altogether."

R. A. Fisher has shown that it is possible to introduce symmetric functions which do simplify the resulting formula appreciably. It is the purpose of this section to introduce an additional symmetric function which simplifies the resulting formulae to a much greater extent. It is admitted that this function does not have all the properties (such as invariance with respect to change of origin) possessed by the Thiele and Fisher functions, but it does not have the property of making the resulting formulae simple. It also has the advantage that $\mu(h'_r) = \mu'(h'_r)$.

The basic idea is to find a sample moment function whose expected value is 0. A first attempt, placing every $b = 0$, is of no avail since every a is also equal to 0 and there is no function. A second attempt is based on the idea of finding the function h whose expected value is $\mu_1'^r$. If the universe is assumed to be measured about its mean, as is conventional, it follows at once that $\mu'_1 = 0$ and $\mu'_1(h_r) = 0$ so that

$$\mu_{\mu\nu}(h'_{r_1}, h'_{r_2}) = \mu'_{\mu\nu}(h'_{r_1}, h'_{r_2}).$$

This function then has the property that its moments about a fixed point and its moments are identical.

In order to discover its expansion in terms of power sums, we note

$$\mu'_1(h'_r) = \mu_1'^r$$

and it follows at once by comparison with $\mu'_1(f_r)$ in section 16 that $b_{1^r} = \frac{1}{n^{(r)}}$ and $b_{p_1^{\pi_1} \cdots p_s^{\pi_s}} = 0$ in all other cases. The a's are determined in the usual way. Thus

$$a_2 = b_2 - b_{11} = -\frac{1}{n(n-1)}$$

$$a_{11} = b_{11} = \frac{1}{n(n-1)}$$

so that

$$h'_2 = -\frac{1}{n(n-1)}[(2) - (1)(1)].$$

Similarly

$$h_3' = \frac{1}{n^{(3)}} [2(3) - 3(2)(1) + (1)^3]$$

$$h_4' = -\frac{1}{n^{(4)}} [6(4) - 8(3)(1) - 3(2)(2) + 6(2)(1)(1) - (1)^4]$$

and in general

$$h_r' = \frac{(-1)^{r-1}}{n^{(r)}} \left\{ \sum (-1)^{\rho-1} [(p_1 - 1)!]^{\pi_1} [(p_2 - 1)!]^{\pi_2} \cdots [(p_s - 1)!]^{\pi_s} \begin{pmatrix} 1^r \\ p_1^{\pi_1} \cdots p_s^{\pi_s} \end{pmatrix} (p_1)^{\pi_1} \cdots (p_s)^{\pi_s} \right\}.$$

In order to show the simple form in which results can be given we substitute the values of the b's in the results obtained above. Not only does $\mu_1'(h_r') = 0$, but by section 11

$$\lambda_2(h_2') = \mu_2(h_2') = \mu_2'(h_2') = \frac{2}{n(n-1)} \mu_2^2$$

$$\lambda_{11}(h_3', h_2') = \mu_{11}(h_3', h_2') = \mu_{11}'(h_3', h_2') = 0$$

$$\lambda_2(h_3') = \mu_2(h_3') = \mu_2'(h_3') = \frac{6}{n(n-1)(n-2)} \mu_2^3$$

while from section 15

$$\lambda_{21}(h_3', h_2') = \mu_{21}(h_3', h_2') = \mu_{21}'(h_3', h_2') = \frac{36\,\mu_3^2\mu_2}{n^2(n-1)^2(n-2)} + \frac{36(n-3)\,\mu_2^4}{n^2(n-1)^2(n-2)}.$$

It is to be noticed that these formulae contain very few terms and that the terms themselves involve very low moments of the universe. This simplicity has been attained without making any assumption such as normality, regarding the nature of the universe.

20. **Table of Values of b for Different Functions When $r < 6$.** This process of defining functions by means of expected values could be extended indefinitely. Perhaps it has been applied to enough functions to suggest the breadth of the applicability of the theory developed in Part I and Part III.

As the b's are the quantities which are used in the formulae I have provided Table III giving their values for the six functions, m_r', m_r, l_r, k_r, h_r, h_r' when $r = 1, 2, 3, 4, 5$. When the a's are known, the b's are computed from them according to the formulae of section 16.

TABLE III

Values of the b's for $r \leqq 5$

Num. coef.	b	m'_r	m_r	l_r	k_r	h_r	h'_r
1	b_1	$\frac{1}{n}$	$\frac{1}{n}$	$\frac{1}{n}$	$\frac{1}{n}$	$\frac{1}{n}$	$\frac{1}{n}$
1	b_2	$\frac{1}{n}$	$\frac{n-1}{n^2}$	$\frac{n-1}{n^2}$	$\frac{1}{n}$	$\frac{1}{n}$	0
1	b_{11}	0	$-\frac{1}{n^2}$	$-\frac{1}{n^2}$	$-\frac{1}{n^{(2)}}$	$-\frac{1}{n^{(2)}}$	$\frac{1}{n^{(2)}}$
1	b_3	$\frac{1}{n}$	$\frac{(n-1)(n-2)}{n^3}$	$\frac{(n-1)(n-2)}{n^3}$	$\frac{1}{n}$	$\frac{1}{n}$	0
3	b_{21}	0	$-\frac{(n-2)}{n^3}$	$-\frac{n-2}{n^3}$	$-\frac{1}{n^{(2)}}$	$-\frac{1}{n^{(2)}}$	0
1	b_{111}	0	$\frac{2}{n^3}$	$\frac{2}{n^3}$	$\frac{2}{n^{(3)}}$	$\frac{2}{n^{(3)}}$	$\frac{1}{n^{(3)}}$
1	b_4	$\frac{1}{n}$	$\frac{(n-1)(n^2-3n+3)}{n^4}$	$\frac{(n-1)(n^2-6n+6)}{n^4}$	$\frac{1}{n}$	$\frac{1}{n}$	0
4	b_{31}	0	$-\frac{(n^2-3n+3)}{n^4}$	$-\frac{(n^2-6n+6)}{n^4}$	$-\frac{1}{n^{(2)}}$	$-\frac{1}{n^{(2)}}$	0
3	b_{22}	0	$\frac{2n-3}{n^4}$	$-\frac{(n^2-4n+6)}{n^4}$	$-\frac{1}{n^{(2)}}$	0	0
6	b_{211}	0	$\frac{n-3}{n^4}$	$\frac{2(n-3)}{n^4}$	$\frac{2}{n^{(3)}}$	$\frac{1}{n^{(3)}}$	0
1	b_{1111}	0	$-\frac{3}{n^4}$	$-\frac{6}{n^4}$	$-\frac{6}{n^{(4)}}$	$-\frac{3}{n^{(4)}}$	$\frac{1}{n^{(4)}}$
1	b_5	$\frac{1}{n}$	$\frac{(n-1)(n-2)(n^2-2n+2)}{n^5}$	$\frac{(n-1)(n-2)(n^2-12n+12)}{n^5}$	$\frac{1}{n}$	$\frac{1}{n}$	0
5	b_{41}	0	$-\frac{(n^3-4n^2+6n-4)}{n^5}$	$-\frac{(n^3-14n^2+36n-24)}{n^5}$	$-\frac{1}{n^{(2)}}$	$-\frac{1}{n^{(2)}}$	0
10	b_{32}	0	$\frac{n^2-4n+4}{n^5}$	$-\frac{(n^3-8n^2+24n-24)}{n^5}$	$-\frac{1}{n^{(2)}}$	0	0
10	b_{311}	0	$\frac{n^2-3n+4}{n^5}$	$\frac{2n^2-18n+24}{n^5}$	$\frac{2}{n^{(3)}}$	$\frac{1}{n^{(3)}}$	0
15	b_{221}	0	$-\frac{2(n-2)}{n^5}$	$\frac{2n^2-12n+24}{n^5}$	$\frac{2}{n^{(3)}}$	0	0
10	b_{2111}	0	$-\frac{n-4}{n^5}$	$-\frac{6(n-4)}{n^5}$	$-\frac{6}{n^{(4)}}$	$-\frac{1}{n^{(4)}}$	0
1	b_{11111}	0	$\frac{4}{n^5}$	$\frac{24}{n^5}$	$\frac{24}{n^{(5)}}$	$\frac{4}{n^{(5)}}$	$\frac{1}{n^{(5)}}$

Part III. Combinatory Methods

21. **Partition Representation of Expected Value of f Functions.** The formulae

$$\mu_1'(f_1) = b_1 n \mu_1'$$

$$\mu_1'(f_2) = b_2 n \mu_2' + b_{11} n(n-1) \mu_1'^2$$

$$\mu_1'(f_3) = b_3 n \mu_3' + 3b_{21} n(n-1) \mu_2' \mu_1' + b_{111} n(n-1)(n-2) \mu_1'^3$$

$$\mu_1'(f_4) = b_4 n \mu_4' + 4b_{31} n(n-1) \mu_2' \mu_1' + 3b_{22} n(n-1) \mu_2'^2 + 6b_{211} n(n-1)(n-2) \mu_2' \mu_1'^2 + b_{1111} n^{(4)} \mu_1'^4$$

are "synthetically" given by the column partitions

1				
2	1			
	1			
3	2	1		
	1	1		
		1		
4	3	2	2	1
	1	2	1	1
			1	1
				1

The partition parts represent both the subscripts of the moments and the subscripts of the b's. If ρ indicates the number of parts, the n multiplier is $n^{(\rho)}$. The numerical coefficient is obtained by taking the sum of the entries in the column (the weight) and dividing it by the factorials of all entries times the factorials of all repeated entries as indicated by

$$\begin{pmatrix} 1^r \\ p_1^{\pi_1} p_2^{\pi_2} \cdots p_s^{\pi_s} \end{pmatrix} = \frac{r!}{(p_1!)^{\pi_1} (p_2!)^{\pi_2} \cdots (p_s!)^{\pi_s} \pi_1! \, \pi_2! \cdots \pi_s!}.$$

The translation from the synthetic partition form to the expanded form is accelerated if the coefficients are known. These are provided in the following partition representation of the formula for $\mu_1'(f_r)$ when $r \leqq 8$ and the results are expressed in terms of the moments of the universe

$\mu_1(f_1)$:	0
$\mu_1'(f_2)$:	1
	2
$\mu_1'(f_3)$:	1
	3

$\mu_1'(f_4)$:	**1**	**3**					
	4	2					
		2					
$\mu_1'(f_5)$:	**1**	**10**					
	5	3					
		2					
$\mu_1'(f_6)$:	**1**	**15**	**10**	**15**			
	6	4	3	2			
		2	3	2			
				2			
$\mu_1'(f_7)$:	**1**	**21**	**35**	**105**			
	7	5	4	3			
		2	3	2			
				2			
$\mu_1'(f_8)$:	**1**	**28**	**56**	**35**	**210**	**280**	**105**
	8	6	5	4	4	3	2
		2	3	4	2	3	2
					2	2	2
							2

The proper formula can be stated immediately from its synthetic representation. Thus for example

$$\mu_1'(f_6) = b_6 n\mu_6 + 15b_{42}n(n-1)\mu_4\mu_2 + 10b_{33}n(n-1)\mu_3^2 + 15b_{222}n(n-1)(n-2)\mu_2^3.$$

22. **Partition Representation of the Expected Value of a Product of f Functions.** Two column partitions may be used similarly to represent the expected values of the products of two f's, three column partitions for the expected value of the triple product, etc. In order to obtain all terms it is only necessary to combine every partition of each f in every possible way. The synthetic representation of $E(m_2, m_1)$ is

1	**1**	**2**	**1**
21	20	11	10
	01	10	10
			01

The sum of the entries in each row indicates the proper moment while the number of rows indicates the number of parts as in the preceding section. The n coefficient associated with a ρ rowed partition is then $n^{(\rho)}$. The b coefficient is indicated by the columnar entries. Thus

$$\mu_{11}'(f_2, f_1 = b_2b_1n\mu_3' + [b_2b_1 + 2b_{11}b_1]n(n-1)\mu_2'\mu_1' + b_{11}b_1n(n-1)(n-2)\mu_1'^3.$$

We verify this by the algebraic method

$$\begin{aligned}
\mu'_{11}(f_2, f_1) &= E\{[a_2(2) + a_{11}(1)(1)][a_1(1)]\} \\
&= E[a_2a_1(2)(1) + a_{11}a_1(1)^3] \\
&= a_2a_1[n\mu'_3 + n(n-1)\mu'_2\mu'_1 \\
&\qquad + a_{11}a_1[n\mu'_3 + 3n(n-1)\mu'_2\mu'_1 + n(n-1)(n-2)\mu'^3_1] \\
&= (a_2 + a_{11})a_1n\mu'_3 + (a_2 + a_{11})a_1n(n-1)\mu'_2\mu'_1 \\
&\qquad + 2a_{11}a_1\mu'_2\mu'_1 + a_{11}a_1n(n-1)(n-2)\mu'^3_1 \\
&= b_2b_1n\mu'_3 + b_2b_1n(n-1)\mu'_2\mu'_1 + 2b_{11}b_1n(n-1)\mu'_2\mu'_1 \\
&\qquad + b_{11}b_1n(n-1)(n-2)\mu'^3_1
\end{aligned}$$

as indicated.

It thus appears that the partition representation is a mnemonic device for indicating the solution as obtained by the algebraic method. A more formal justification is based upon the property that if

$$E(f_2) = b_2(2) + b_{11}(1)(1) \quad \text{and} \quad E(f_1) = b_1(1)$$

then $E(f_2, f_1)$ can be obtained by a symbolic multiplication of $b_2(2) + b_{11}(1)(1)$ by $b_1(1)$ where the b's are multiplied but the power sums are collected in all possible ways. Thus

$$E(f_2, f_1) = b_2b_1[(3) + (2)(1)] + b_{11}b_1[2(2)(1) + (1)^3]$$

which gives

$$E(f_2, f_1) = b_2b_1n\mu'_3 + b_2b_1n(n-1)\mu'_2\mu'_1 + 2b_{11}b_1n(n-1)\mu'_2\mu'_1 + b_{11}b_1n^{(3)}\mu'^3_1$$

as before.

This symbolic multiplication is generally true and serves as the real algebraic justification of the partition representation. It will be established in a later paper dealing with the more general case of a finite population. The general type of partition analysis has been used previously by Fisher (3) and Georgescu (4). Each has established it through analytic rather than algebraic means.

23. **Determination of the Coefficients.** Methods of determining the numerical coefficient have previously been given by such authors as Fisher (3), Wishart (5) (7) and Georgescu (4). If the f's are of different weight, the coefficients of any partition (an interchange of rows is not looked upon as changing the partition) is given by writing in the numerator the factorials of the different r's and in the denominator the factorials of all the different entries and the factorials of all repeated rows. Thus the coefficient of

$$\begin{matrix} 210 \\ 111 \\ 111 \end{matrix} \quad \text{is} \quad \frac{4!\,3!\,2!}{2!\,(1!)^7\,2!} = 72.$$

In case two or more functions have the same weight additional equivalent partitions are formed by interchange of columns. The reader is referred to the above papers for rules for determining the coefficients in the more involved cases though the coefficients are presented for all the two way partitions of the next section.

An alternative method of finding the coefficients is that given by C. C. Craig (2, 24–25) since it appears that the symbolic formulae used in the present paper are essentially his formulae for ν's in terms of λ's. For example his formula for ν_{44} (2, 22) is given symbolically by the formula for 44 in the next section. The only difference revealed is that the subscripts of the λ's are read by rows rather than by columns and that they are sometimes interchanged. The more precise formulization is needed for the present interpretation although it is not needed for Prof. Craig's purpose.

A third method utilizes the symbolic multiplication process stated in section 22. Subscripts of the b's are used to indicate which power sums are collected. Thus $[b_2(2) + b_{11}(1)(1)]^2$ gives

$$\underline{b_2b_2(4) + b_{20}b_{02}(2)(2)} + \underline{2[2b_{20}b_{11}(3)(1) + b_{200}b_{011}(2)(1)(1)]} + \underline{2b_{11}b_{11}(2)(2)} + \underline{4b_{110}b_{101}(2)(1)(1) + b_{1100}b_{0011}(1)(1)(1)(1)}$$

where the underscored terms indicate the products given by $[b_2(2)]^2$, $2[b_2(2)][b_{11}(1)(1)]$, and $[b_{11}(1)(1)]^2$ respectively. This is represented by

1	**1**	**4**	**2**	**2**	**4**	**1**
22	20	21	20	11	11	10
	02	01	01	11	10	10
			01		01	01
						01
—	—			—		

The underscored terms are the only ones remaining when $\mu_1' = 0$.

This method is especially useful when a large number of formulae are to be computed, as in the next section.

24. **The Partition Representation of Formulae of Total Weight $\leqq$ 8.** The partition representation of $\mu_1'(f_r)$ when $r \leqq 8$ are given in section 21. The partition representation of the remaining formulae of total weight $\leqq 8$, which do not contain unit parts, are given below.

22	**1**	**1**	**2**
	22	20	11
		02	11

32	**1**	**1**	**3**	**6**
	32	30	12	21
		02	20	11

42	**1**	**1**	**8**	**6**	**4**	**6**	**3**	**12**					
	42	40	31	22	30	21	20	20					
		02	11	20	12	21	20	11					
							02	11					
33	**1**	**6**	**9**	**1**	**9**	**9**	**6**						
	33	31	22	30	21	20	11						
		02	11	03	12	11	11						
						02	11						
222	**1**	**3**	**12**	**6**	**4**	**1**	**6**	**8**					
	222	220	211	201	111	200	200	110					
		002	011	021	111	020	011	011					
						002	011	101					
52	**1**	**1**	**10**	**10**	**5**	**10**	**20**	**10**	**20**	**15**	**60**		
	52	50	41	32	40	22	31	30	30	12	21		
		02	11	20	12	30	21	20	11	20	20		
								02	11	20	11		
43	**1**	**3**	**12**	**6**	**1**	**4**	**12**	**18**	**12**	**3**	**18**	**36**	**36**
	43	41	32	23	40	13	31	22	30	03	21	12	21
		02	11	20	03	30	12	21	11	20	20	20	11
									02	20	02	11	11
322	**1**	**2**	**4**	**12**	**3**	**1**	**4**	**6**	**12**	**12**			
	322	320	311	221	122	022	301	220	121	211			
		002	011	101	200	300	021	102	201	111			
	1	**2**	**6**	**12**	**12**	**12**	**24**	**12**	**24**				
	300	300	102	021	201	111	210	120	111				
	020	011	020	101	020	011	101	101	101				
	002	011	200	200	101	200	011	101	110				
62	**1**	**1**	**12**	**15**	**6**	**30**	**20**	**15**	**20**				
	62	60	51	42	50	41	32	40	31				
		02	11	20	12	21	30	22	31				
	15	**30**	**120**	**45**	**10**	**60**	**120**	**90**		**15**	**90**		
	40	40	31	22	30	30	30	21		20	20		
	20	11	20	20	30	12	21	21		20	20		
	02	11	11	20	02	20	11	20		20	11		
										02	11		

53	**1**	**3**	**15**	**10**	**1**	**15**	**30**	**10**	**5**	**30**		
	53	51	42	33	50	41	32	23	40	31		
		02	11	20	03	12	21	30	13	22		
	15	**60**	**90**	**15**	**30**	**10**	**30**	**60**	**90**	**90**	**45**	**60**
	40	31	22	13	31	30	30	30	12	21	20	20
	11	11	20	20	20	03	21	12	21	21	20	11
	02	11	11	20	02	20	02	11	20	11	11	11
											02	11
44	**1**	**12**	**16**	**8**	**48**	**1**	**16**	**18**				
	44	42	33	41	32	40	31	22				
		02	11	03	12	04	13	22				
	6	**96**	**36**	**72**	**48**	**16**	**72**	**144**	**9**	**72**	**24**	
	40	31	22	22	30	30	21	21	20	20	11	
	02	11	20	11	12	03	21	12	20	11	11	
	02	02	02	11	02	11	02	11	02	11	11	
									02	02	11	
422	**1**	**2**	**4**	**16**	**6**	**4**	**8**	**4**	**24**	**16**		
	422	420	411	321	222	401	320	122	212	311		
		002	011	101	200	021	102	300	210	111		
	1	**16**	**6**	**12**								
	400	310	220	211								
	022	112	202	211								
	1	**2**	**16**	**32**	**12**	**3**	**24**	**24**	**48**	**48**		
	400	400	310	310	202	022	211	220	211	121		
	020	011	110	101	200	200	200	101	101	200		
	002	011	002	011	020	200	011	101	110	101		
	8	**16**	**12**	**24**	**12**	**16**	**48**	**96**	**24**	**24**		
	300	300	210	021	120	300	201	210	111	210		
	120	021	210	201	102	111	120	111	111	201		
	002	101	002	200	200	011	101	101	200	011		
	3	**24**	**6**	**48**	**24**							
	200	200	200	200	110							
	200	110	200	110	110							
	020	110	011	101	101							
	002	002	011	011	101							

332	**1**	**1**	**9**	**12**	**6**	**2**	**18**	**18**	**6**	**12**
	332	330	222	321	312	302	212	221	320	311
		002	110	011	020	030	120	111	012	021

2	**9**	**18**	**6**	
301	220	211	310	
031	112	121	022	

9	**18**	**6**	**12**	**12**	**18**	**9**	**72**	**18**	**36**
220	220	310	301	310	202	112	211	112	211
110	101	020	020	011	110	200	110	110	101
002	011	002	011	011	020	020	011	110	020

1	**6**	**12**	**9**	**18**	**36**	**36**	**18**	**36**	**72**	**36**
300	300	300	210	210	210	201	201	210	210	111
030	012	021	120	102	012	111	021	101	111	111
002	020	011	002	020	110	020	110	021	011	110

9	**18**	**36**	**6**	**36**
200	200	200	110	110
110	101	110	110	110
020	011	011	110	101
002	020	011	002	011

2222	**1**	**4**	**24**	**24**	**32**	**3**	**24**	**8**
	2222	2220	2211	2201	2111	2200	2011	1111
		0002	0011	0021	0111	0022	0211	1111

6	**12**	**48**	**96**	**48**
2200	2200	2011	2011	1111
0020	0011	0011	0101	1100
0002	0011	0200	0110	0011

24	**48**	**96**	**16**	**48**	**16**	**32**
2001	2010	2100	0111	1011	1011	0111
0201	0201	0111	0111	1110	0111	1101
0020	0011	0011	2000	0101	1100	1010

1	**12**	**32**	**12**	**48**
2000	2000	2000	1100	1100
0200	0200	0101	1100	0110
0020	0011	0110	0011	0011
0002	0011	0011	0011	1001

25. **The Formulae for the Sample Moments about a Fixed Point in Terms of the Moments of the Universe.** The partitions of section 21 and section 24 can be immediately interpreted to give the formulae for the moments of the sample function. For example

$$\mu'_{11}(f_3, f_2) = b_3b_2n\mu_5 + (b_3b_2 + 3b_{21}b_2 + 6b_{21}b_{11})n(n-1)\mu_3\mu_2$$

and the value of $\mu'_{21}(f_3, f_2)$ as given in section 15 can be read by inspection. The value of the b's are to be inserted for any specific function. The coefficient of μ_2^3 in the expansion of $\mu'_3(f_2)$ is

$$(b_2^3 + 6b_2b_{11}^2 + 8b_{11}^3)n(n-1)(n-2).$$

In case $f_2 = m_2$, $b_2 = \dfrac{n-1}{n^2}$, and $b_{11} = \dfrac{-1}{n^2}$ so that the coefficient is

$$\frac{(n-1)(n-2)(n^3 - 3n^2 + 9n - 15)}{n^5}$$

as indicated previously by Tchouproff (10, 192) and Church (9, 82).

The partitions of section 21 give the 8 formulae $\mu_{r,(N)}$ which Tchouproff gave (10, 155). In this case $f_r = m'_r$ and every b is 0 except those having single subscripts and these equal $\dfrac{1}{n}$.

The partitions of section 21 give the formulae $\nu_{r,(N)}$ which were given by Tchouproff (10, 186). In this case it is only necessary to take $f_r = m_r$ and to give the b's the proper values. Tchouproff has arranged his results according to decreasing powers of n. As an illustration we derive his result for $\nu_{4,(N)} = \mu'_1(m_4)$. From section 21

$$\mu'_1(f_4) = b_4n\mu_4 + 3b_{22}n(n-1)\mu_2^2$$

and from Table II

$$b_4 = \frac{(n-1)(n^2 - 3n + 3)}{n^4} \quad \text{and} \quad b_{22} = \frac{2n-3}{n^4}$$

so that

$$\mu'_1(m_4) = \left(1 - \frac{4}{n} + \frac{6}{n^2} - \frac{3}{n^3}\right)\mu_4 + \left(\frac{6}{n} - \frac{15}{n^2} + \frac{9}{n^3}\right)\mu_2^2$$

$$= \mu_4 + \frac{1}{n}(6\mu_2^2 - 4\mu_4) - \frac{1}{n^2}(15\mu_2^2 - 6\mu_4) + \frac{1}{n^3}(9\mu_2^2 - 3\mu_4)$$

as indicated by him.

The partitions of section 24 also give formulae which have appeared before. For example the partitions

1	**1**	**2**
22	20	11
	02	11

which symbolize the formula

$$\mu_2'(f_2) = b_2^2 n\mu_4 + (b_2^2 + 2b_{11}^2)n(n-1)\mu_2^2$$

become

$$\mu_2'(m_2) = \frac{(n-1)}{n^3}[(n-1)\mu_4 + (n^2 - 2n + 3)\mu_2^2]$$

which was early derived by "Student" (8, 3) and Tchouproff (10, 192). Similarly the partitions of 222 and 2222 give the formula for and $\mu_3'(m_2)$ and $\mu_4'(m_2)$ which were given by Tchouproff (10, 192–193) and Church (9, 82).

Sections 21 and 24 can then be used to write the moments about a fixed point of a sample function in terms of the moments of the universe. In the case of new functions the b's must first be determined. Formulae involving unit columnar partitions are not included. If the formulae were desired in terms of moments about a fixed point of the universe, it would be necessary to write in addition all possible partitions. See for example the last formula of section 23.

26. **The Formulae For Moments of Any Sample Function in Terms of Moments of the Universe.** The partitions of sections 21 and 24 are also useful in writing the formulae for the moments of the sample moments. It is necessary to make the usual adjustments in changing from moments about a fixed point to moments:

$$\mu_2(f_r) = \mu_2'(f_r) - \mu_1'^2(f_r)$$

$$\mu_{11}(f_{r_1}, f_{r_2}) = \mu_{11}'(f_{r_1}, f_{r_2}) - \mu_{10}'(f_{r_1}, f_{r_2})\mu_{01}'(f_{r_1}, f_{r_2}).$$

The particular two way partitions which are involved in this adjustment are immediately recognizable. They are the ones which have an entry which is the only entry in the row and in the column in which it is. Thus $\begin{matrix}\mathbf{3}\\220\\002\end{matrix}$ gives one of the terms contributing to $\mu_2'(f_2)\,\mu_1'(f_2)$. In addition its coefficient is the same, if sign is not considered, as the coefficient of $\mu_2'(f_2)\,\mu_1'(f_2)$ in the expansion of $\mu_3(f_2)$ in terms of moments of f_2. This has to be true since each is the number of ways of forming $\begin{matrix}220\\002\end{matrix}$. And so in general the remaining function of n accompanying this adjustment is the product of the coefficient associated with 22 and that associated with 2. The sign is plus when odd numbers of moments are multiplied and minus when even numbers of moments are multiplied. Hence $\begin{matrix}\mathbf{3}\\220\\002\end{matrix}$ contributes $-3n^2b_2^3$ to the adjustment to moments and the total contribution of $\begin{matrix}\mathbf{3}\\220\\002\end{matrix}$ to the value of $\mu_3(f_2)$ is $3b_2^3[n(n-1) - n^2] = -3b_2^3 n$. More

extensive study leads to the following general method of using the formulae of section 24.

A. Write the coefficient of every two way partition according to section 25.

B. Block off each single entry by drawing a line through its row and column. For example

$$\begin{array}{c} \mathbf{6} \\ 2200 \\ 0020 \\ 0002 \end{array}$$

The resulting partitions, 22, 2, 2 are called component parts.

C. Form new partitions by eliminating component parts one at a time, two at a time, three at a time, etc. from the original partition in all possible ways.

D. Form the coefficient of the resulting parts according to the methods of section 25. Multiply by $(-1)^{s-1}$ where s is the number of resulting parts. The values of b will not change.

E. Multiply in addition by $s - 1$ when the component parts are all taken separately.

As an example we find the contribution of the partition $\begin{array}{c} \mathbf{6} \\ 2200 \\ 0020 \\ 0002 \end{array}$ to the value of $\mu_4(f_2)$. It gives

$$6b_2^4[n(n-1)(n-2) - 3n^2(n-1) + 2n^3]\mu_4\mu_2\mu_2 = 12nb_2^4\mu_4\mu_2^2.$$

Similarly $\begin{array}{l} \mathbf{1} \text{ contributes} \\ 2000 \\ 0200 \\ 0020 \\ 0002 \end{array}$

$$b_2^4[n^{(4)} - 4nn^{(3)} + 6n^3(n-1) - 3n^4]\mu_2^4 = 3b_2^4(n-2)\mu_2^4.$$

We use the method in finding the coefficient of μ_2^3 in the expansion of $\mu_3(m_2)$. We find first the coefficient of μ_2^3 in the expansion of $\mu_3(f_2)$. It is indicated by the partitions

1	**6**	**8**
200	200	110
020	011	011
002	011	101

so that the coefficient of μ_2^3 is

$$b_2^3[n(n-1)(n-2) - 3n^2(n-1) + 2n^3] + 6b_2b_{11}^2[n(n-1)(n-2) - n^2(n-1)]$$
$$+ 8b_{11}^3n(n-1)(n-2) = b_2^3(2n) + 6b_2b_{11}^2(-2n^2 + 2n)$$
$$+ 8b_{11}^3n(n-1)(n-2).$$

When $b_2 = \dfrac{n-1}{n^2}$ and $b_{11} = \dfrac{-1}{n^2}$ this becomes $\dfrac{2(n-1)(n^2-12n+15)}{n^5}$ as previously given by such authors as Tchouproff (10, 194), Church (9, 82), Carver (Richardson) (11, 271).

The general Tchouproff-Church formulae for the third and fourth moments of the variance may be written out in this way as may many other moment formulae which have not been printed.

27. **The Thiele Moments of the Sample Function in Terms of the Moments of the Universe.** It is possible also to write the Thiele moments of the sample function in terms of the moments of the universe. The technique is very similar to that of the previous section. The basis of the transformation is now the formula for Thiele moments in terms of moments about a fixed point rather than moments in terms of moments about a fixed point. The results are the same as those of the last section when a double or a triple product of f's is involved, but they differ with the introduction of a larger number of products. The partitions having component parts are broken up into these component parts as before but the parts are combined in all possible ways. Multipliers are determined as before with the exception that there is a multiplication by $(-1)^{s-1}(s-1)!$ where s is the number of resultant parts. Thus the

$$\text{term } \begin{matrix}2000\\0200\\0020\\0002\end{matrix} \text{ contributes } b_2^4[n^{(4)} - 4nn^{(3)} - 3n^2(n-1)^2 + 12n^3(n-1) - 6n^4]\mu_2^4 =$$

$-6b_2^4 n\mu_2^4$ to the value of $\lambda_4(f_2)$.

28. **The Moments About a Fixed Point of the Sample Function in Terms of the Thiele Moments of the Universe.** We return to the problem of section 25, only we wish to express the results in terms of the Thiele moments of the universe. We must use the formulae of section 12.

$$\mu_r = \sum \binom{1^r}{p_1^{\pi_1} \cdots p_s^{\pi_s}} (\lambda_{p_1})^{\pi_1} \cdots (\lambda_{p_s})^{\pi_s}$$

where $p_i \neq 1$.

Thus μ_r will contribute to all partitions of r and inversely the contributions to a given partition are composed only of these terms which are obtained by combining the different elements of the partition. Since the numerical coefficient in the expansion of μ_r is the number of ways in which the r units can be collected to form the partition, it follows at once that the complete λ coefficient can be obtained by grouping the parts of the partition in all possible ways, determining the coefficient of each according to the methods of section 25, and adding. In this way the formulae of section 21 can be used to give expansions in terms of partition moments. For example the representation of $\mu_1'(f_6)$

1	**15**	**10**	**15**
6	4	3	2
	2	3	2
			2

gives at once

$$b_6 n\lambda_6 + 15[b_6 n + b_{42} n(n-1)]\lambda_4\lambda_2 + 10[b_6 n + b_{33} n(n-1)]\lambda_3^2 + 15[b_6 n + 3b_{42} n(n-1) + b_{222} n(n-1)(n-2)]\lambda_2^3.$$

The partitions of section 21 can be made to give the formula $\mu_1'(l_r)$ which were given by Thiele (1, 45–46). For example the formula for $\mu_1'(f_4)$ is indicated by

1	**3**
4	2
	2

so that

$$\mu_1'(f_4) = b_4 n\lambda_4 + 3[b_4 n + b_{22} n(n-1)]\lambda_2^2$$

and since

$$b_4 = \frac{(n-1)(n^2-6n+6)}{n^4} \quad \text{and} \quad b_{22} = \frac{2n-3}{n^4}$$

$$\mu_1'(l_4) = \frac{(n-1)(n^2-6n+6)\lambda_4}{n^3} - \frac{6(n-1)\lambda_2^2}{n^2},$$

which agrees with the result as given by him (1, 45).

The two way partitions of section 24 can be used similarly. This device for changing to the λ's is due to the ingenuity of R. A. Fisher who applied it to the case where $f_r = k_r$.

As an illustration we write from section 24 the value of $\mu_2'(f_2)$ in terms of λ's. The partition representation

1	**1**	**2**
22	20	11
	02	11

gives at once

$$b_2^2 n\lambda_4 + [b_2^2 n + b_2^2 n(n-1)]\lambda_2^2 + 2[b_2^2 n + b_{11}^2 n(n-1)]\lambda_2^2$$

which agrees with the result of section 12. The other illustrations of that section may be written out similarly.

As a final illustration of this technique we find the coefficient of λ_4^2 in the expansion of $\mu_{21}'(f_3, f_2)$. The partitions are

2	**9**	**18**	**6**
301	220	211	310
031	112	121	022

and the coefficient is

$$2[b_3^2b_2n + b_3^2b_{11}n(n-1)] + 9[b_3^2b_2n + b_{21}^2b_2n(n-1)]$$
$$+ 18[b_3^2b_2n + b_{21}^2b_{11}n(n-1)] + 6[b_3^2b_2n + b_3b_{21}b_2n(n-1)].$$

If the b's are inserted to form the k's, the first and last terms become 0 and the others give $\dfrac{27n-45}{n(n-1)^2}$. This agrees with the value as given by R. A. Fisher (3, 208).

29. **The Moments of the Sample Function in Terms of the Thiele Moments of the Universe.** The partition representations of section 21 and section 24 can be used similarly to write formulae for the moments of the sample function in terms of the Thiele moments of the universe. It is only necessary to use the general plan of section 26, but to write the coefficient of every resulting partition according to the method of section 28. For example the partition

$$\begin{array}{c} 2000 \\ 0200 \\ 0020 \\ 0002 \end{array}$$

gives the coefficient

$$b_2^4[n + 4n^{(2)} + 3n^{(2)} + 6n^{(3)} + n^{(4)}] - 4b_2^4[n^2 + 3n^2(n-1) + n^2(n-1)(n-2)]$$
$$+ 6b_2^4[n^3 + n^3(n-1)] - 3b_2^4n^4 = b_2^4[n^4 - 4n^4 + 6n^4 - 3n^4] = 0.$$

30. **The Thiele Moments of the Sample Function in Terms of the Thiele Moments of the Universe.** The partition representations of section 21 and section 24 can also be interpreted to give the Thiele moments of the sample function in terms of the Thiele moments of the universe. The scheme is similar to that of section 29 except that the formulae for changing to Thiele moments are used as in section 27. For example the partition $\begin{array}{c} 2000 \\ 0200 \\ 0020 \\ 0002 \end{array}$ has now associated with it

$$b_2^4[n + 4n^{(2)} + 3n^{(2)} + 6n^{(3)} + n^{(4)}] - 4b_2^4[n^2 + 3n^2(n-1) + n^2(n-1)(n-2)]$$
$$- 3b_2^4n^2(n-1)^2 + 12b_2^4[n^3 + n^3(n-1)] - 6b_2^4n^4 = 0.$$

The application of this method enables one to write the formulae of section 13 (and others which they typify) with relative ease. It is now possible to complete the task left unfinished in section 15. We do not take the space necessary to write all the terms of $\lambda_{21}(f_3, f_2)$ since the lengthy expression can be obtained quite readily from the representation of section 24. One term, say the coefficient of $\lambda_6\lambda_2$, is represented by

1	9	12	6
330	222	321	312
002	110	011	020

and gives

$$9[b_3^2b_2n + b_{21}^2b_2n(n-1)] + 12[b_3^2b_2n + b_3b_{21}b_{11}n(n-1)] + 6[b_3^2b_2n + b_3b_{21}b_2n(n-1)]$$

which becomes $\frac{21}{n(n-1)}$ when $b_3 = b_2 = \frac{1}{n}$ and $b_{21} = b_{11} = \frac{-1}{n(n-1)}$. This agrees with the result given by R. A. Fisher (3, 209).

For simplicity of form it is logical to use this formulization of results, Thiele moments in terms of Thiele moments, and it has been used by Thiele (1), Craig (2), Fisher (3) and Georgescu (4). They however have used different sample moment functions. Thiele and Georgescu used the Thiele moments of the sample, Craig and Georgescu the moments while Fisher introduced the k function.

The present discussion deals with the corresponding partition moments of any rational integral isobaric moment function of the sample. The results indicated here give many of the results of the previous authors as special cases. For example the symbolic formula 44 of section 24 gives the $m^7\lambda_2(\mu_4)$ of Thiele (1, 45), the $S_{02}(\nu_2, \nu_4)$ of Craig (2, 57), the $\kappa(44)$ of R. A. Fisher (3, 210) as special cases when the formula 44 is given the interpretation of this section.

Some may prefer the Craig attack (2, 21–35) to the partition method. It should be noted that the formulae of sections 21 and 24 can be used in place of part of the Craig method. Thus his formulae (2, 22)

$$\nu_{80} = \lambda_{80} + 28\,\lambda_{60}\lambda_{20} + 56\,\lambda_{50}\lambda_{30} + \text{etc.}$$

$$\nu_{44} = \lambda_{44} + (12\,\lambda_{42}\lambda_{02} + 16\,\lambda_{33}\lambda_{11}) + \text{etc.}$$

are immediately obtainable from the symbolic formulae by writing λ's in place of b's and by using row, rather than column, subscripts. It is then necessary to compute the values of $\lambda_{k_1k_2}\cdots$ as given by him (2, 16–17, 40) and to insert in his expansions of $S_{kl}(\nu_m, \nu_n)$ in terms of ν's. For example

$$S_{11}(\nu_3, \nu_2) = \frac{1}{n}[\nu_{50} + (n-1)\nu_{32} - n\nu_{30}\nu_{02}] \qquad (2, 32)$$

and from the symbolic formulae of sections 21 and 24

$$\nu_{50} = \lambda_{50} + 10\lambda_{30}\lambda_{20}$$

$$\nu_{32} = \lambda_{32} + \lambda_{30}\lambda_{02} + 3\lambda_{12}\lambda_{20} + 6\lambda_{21}\lambda_{11}$$

$$\nu_{30} = \lambda_{30}$$

$$\nu_{20} = \lambda_{20}$$

so that

$$S_{11}(\nu_3 , \nu_2) = \frac{1}{n}[\lambda_{50} + (n-1)\lambda_{32} + 9\lambda_{30}\lambda_{20} + (n-1)(6\lambda_{21}\lambda_{11} + 3\lambda_{21}\lambda_{20})] \qquad (2, 30)$$

which agrees with that given by Prof. Craig (aside from an obvious typographical error). The insertion of the values of λ gives the value as indicated by $\lambda_{11}(m_3 , m_2)$ of section 13 and by the first method of the present section.

31. **Special Rules for the Determination of the Coefficients in the Case of the Fisher and Georgescu Analyses.** R. A. Fisher (3) gave a number of simple rules which assist greatly in the determination of the coefficients accompanying the partitions. Georgescu (4) also introduced special rules for the evaluation of the coefficients of the different partitions he used. It is not to be expected that all these rules are applicable in the more general case under present consideration, but the vanishing of such coefficients as that of 2000 leads one to

$$\begin{matrix} 0200 \\ 0020 \\ 0002 \end{matrix}$$

suspect that there might be some rules which are applicable to this general case. A sensible method of procedure is to examine the rules of Fisher and Georgescu and determine if they hold in the more general analysis. The special rules of R. A. Fisher might be given somewhat as follows.

A. If a partition has a column with a single entry, that column may be eliminated and the factor n^{-1} introduced.

B. Any partition having a row with a single entry may be neglected.

C. "We may exclude any partition in which any set of rows is connected to its complementary set by a single column only."

D. In determining the algebraic coefficient of a partition the "pattern" is sufficient and precise entries are not needed. Thus the partitions $\begin{matrix} 21 \\ 11 \end{matrix}$ and $\begin{matrix} 35, \\ 42 \end{matrix}$ although they have different numerical factors, have associated with them the same function of n. This value is indicated by the pattern $\begin{matrix} xx \\ xx \end{matrix}$ which has associated with it the function $\frac{1}{n-1}$. As a result of this property Fisher was able to provide a table (3, 223–226) of useful patterns which is of great assistance in writing the value of the coefficients.

E. Formulae of moments of k functions involving k_1 can be derived from corresponding formulae not involving k_1. "The effect upon the corresponding formula of adding a new unit part to the partition is (1) to modify every term in the formula by increasing the suffix of one of its κ functions by unity in every possible way, and (2) to divide the whole by n." (3, 206).

Two of the important Georgescu rules may be stated.

A′. The numerator function (aside from numerical coefficient) is not altered

if columns are changed to rows and vice versa. Thus the coefficient of s_2^3 in $S(3^2) = \dfrac{6N(N-1)}{(N+1)^3}$ and the coefficient of s_3^2 in $S(2^3)$ is $\dfrac{4N(N-1)}{(N+1)^4}$. Georgescu has replaced n by $N+1$.

B′. All partitions which can be broken up into component parts have coefficients of 0. This is extended to include all partitions which have as component parts other partitions. Thus

$$\begin{matrix} 2100 \\ 1100 \\ 0012 \\ 0034 \end{matrix}$$

has a coefficient 0 as does the equivalent

$$\begin{matrix} 2010 \\ 1010 \\ 0102 \\ 0304 \end{matrix}$$

32. **Special Rules for the Determination of the Coefficients in the More General Case.** In the more general case we have

A. If a partition has a single column with a single entry, c, that column may be eliminated and the value b_c inserted as a multiplier. This is immediately evident since the contribution of that column to each term in the expansion is b_c times its value if the column were eliminated.

B. The coefficient of any partition having an entry which is the only entry in its row and column, is 0.

This rule, which saves considerable labor in that it makes unnecessary the computation of the coefficients of many of the partitions of section 24, is established in this way. Without loss of generality the partition may be represented by

$$\pi_{u+1,\,v+1} = \begin{matrix} c_{11} & c_{12} & c_{13} & \cdots & c_{1v} & 0 \\ c_{21} & c_{22} & c_{23} & \cdots & c_{2v} & 0 \\ c_{31} & c_{32} & c_{33} & \cdots & c_{3v} & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ c_{u1} & c_{u2} & c_{u3} & \cdots & c_{uv} & 0 \\ 0 & 0 & 0 & & 0 & c_{u+1,\,v+1} \end{matrix}$$

and $\pi_{u,\,v}$ may represent the partition containing the first u rows and the first v columns. We determine the coefficient of $\pi_{u+1,\,v+1}$ in terms of the coefficient of $\pi_{u,\,v}$. Consider first any grouping of the u rows of $\pi_{u,\,v}$ into w rows. There will be w corresponding groupings of $\pi_{u+1,\,v+1}$ in which the last row is added, in turn, to each of the w rows and another $w+1$ rowed term in which it is not

added. In each of the first w cases the coefficient by rule A is multiplied by $b_{c_{u+1,v+1}}$. In the case of the $w+1$ rowed partition the coefficient is multiplied by $b_{c_{u+1,v+1}}$, and $n^{(w)}$ is replaced by $n^{(w+1)}$. A final adjustment takes care of the transition from the moment about a fixed point of the sample function to the Thiele moment of the sample function. This adjustment demands the multiplication of the coefficient of $\pi_{u,v}$ by $b_{c_{u+1,v+1}}\, n$ and the subtraction from the sum of the other terms. If B_w is the coefficient of the w rowed form, it follows at once that the corresponding coefficient is

$$B_w b_{c+1,v+1}\,[wn^{(w)} + n^{(w+1)} - n\,n^{(w)}] = 0.$$

This holds for the expansion of any term of $\pi_{u,v}$ and hence the coefficient of $\pi_{u+1,v+1}$ is 0. Of course the argument holds if the partition has more than 2 component parts.

It thus appears that this rule holds not only for k_r and m_r as Fisher and Georgescu have noted, but for f_r.

C. The coefficient of any partition which can be broken into component parts is 0. In this sense a component part is any group of rows or columns which have no entry in common with any other group of rows or columns. It corresponds in matrix language to a matrix which results when one matrix is zero bordered by another matrix although rows and columns may thereafter be interchanged.

The proof of this more general case follows the general line of the simpler case although the reasoning is more complicated. For example the coefficient of

$$\begin{array}{ccccc}
c_{11} & c_{12} \cdots & c_{1v} & 0 & 0 \\
c_{21} & c_{22} \cdots & c_{2v} & 0 & 0 \\
c_{31} & c_{32} \cdots & c_{3v} & 0 & 0 \\
c_{u1} & c_{u2} \cdots & c_{uv} & 0 & 0 \\
0 & 0 \cdots & 0 & c_{u+1,v+1} & c_{u+1,v+2} \\
0 & 0 \cdots & 0 & c_{u+2,v+1} & c_{u+2,v+2}
\end{array}$$

is 0 since any w rowed term of the $\pi_{u,v}$ contributes

$$B_w b_{c_{u+1,v+1}+c_{u+2,v+1}}\, b_{c_{u+2,v+2}+c_{u+2,v+2}}\,[wn^{(w)} + n^{(w+1)} - nn^{(w)}]$$
$$+ B_w b_{c_{u+1,v+1}c_{u+2,v+2}}\, b_{c_{u+1,v+2}c_{u+2,v+2}}\,[w(w-1)\,n^{(w)} + 2wn^{(w+1)} + n^{(w+2)} - n(n-1)\,n^{(w)}] = 0.$$

Other special rules of Fisher and Georgescu do not hold in the general case. Thus Fisher rule B is not generally true since the partitions

$$\begin{array}{ccc} 12 & \text{and} & 22 \\ 30 & & 20 \end{array}$$

have respective algebraic coefficients of $b_4b_2n + b_{31}b_2n(n-1)$ and

$$b_4b_2n + b_{22}b_2n(n-1)$$

and these are not in general equal to 0.

The Fisher rule C is replaced by the somewhat less general C of the present section.

The Fisher rule D is not applicable in the general case. The Fisher rule D is applicable in all cases in which the value of the $b_{p_1^{\pi_1}\cdots p_s^{\pi_s}}$ is completely determined by the number of parts for in this case the particular value of each part is not pertinent. We may say then that the Fisher rule D is applicable to all cases in which $b_{p_1^{\pi_1}p_2^{\pi_2}\cdots p_s^{\pi_s}}$ is a function of ρ, n where ρ is the number of parts. This condition is satisfied by $b_{p_1^{\pi_1}\cdots p_s^{\pi_s}} = \dfrac{(-1)^\rho(\rho-1)!}{n^{(\rho)}}$ and the coefficients are worked out for it in Fisher's paper. The same method is applicable to other functions satisfying the general condition although the values of the coefficients will of course vary with the definition of b.

The Fisher rule E is not applicable to the general case. Its validity, from an algebraic standpoint, depends upon the Fisher property B which is not generally applicable. The Fisher rule E as applied to the more general case gives correct terms but it does not give all the terms. For example the Fisher rule E applied to $\lambda_2(k_2)$ gives

$$\lambda_2(k_2) = \frac{\lambda_4}{n} + \frac{2\lambda_2^2}{n-1}$$

$$\lambda_{21}(k_2, k_1) = \frac{\lambda_5}{n^2} + \frac{4\lambda_3\lambda_2}{n(n-1)}.$$

The application of a corresponding rule to

$$\lambda_2(f_2) = b_2^2n\lambda_4 + 2[b_2^2n + b_{11}n(n-1)]\lambda_2^2$$

would give

$$\lambda_{21}(f_2,f_1) = b_2^2b_1n\lambda_5 + 4[b_2^2b_1n + b_{11}^2b_1n(n-1)]\lambda_3\lambda_2$$

while the correct result is indicated by

1	4	2	4
221	210	201	111
	011	020	110

and is

$$\lambda_{21}(f_2f_1) = b_2^2b_1n\lambda_5 + 4[b_2^2b_1n + b_2b_{11}b_1n(n-1)]\lambda_3\lambda_2 + 2[b_2^2b_1n + b_2^2b_1n(n-1)]\lambda_3\lambda_2$$
$$+ 4[b_2^2b_1n + b_{11}^2b_1n(n-1)]\lambda_3\lambda_2.$$

The difference is due to the vanishing of the two middle terms in the case of the k functions.

The rule B′, which Georgescu found most useful in computing and checking his formulae, is not generally true. It is not even true in the case of the k function, as can be discovered by using it on the list given by R. A. Fisher (3, 210). It is interesting to note that the Georgescu method, while not being able to utilize many of the special rules of the Fisher method, does use this rule which is not in general adaptable to the Fisher method.

33. **Special Rules in the Case of the h' Functions.** Special rules can be worked out for other sample functions. As an illustration we examine the function h_r' which was defined in section 19. It is recalled that $b_{1^\rho} = \frac{1}{n^{(\rho)}}$ and that $b_{p_1^{\pi_1} \cdots p_s^{\pi_s}} = 0$ for all other cases. It follows at once that

A. Any partition having any entry other than unity (or zero) may be neglected.

B. The value of b_{1^ρ} is $\frac{1}{n^{(\rho)}}$.

As an illustration we write the value $\lambda_{21}(h_3', h_2')$. From the partitions of section 24 we select

36		**36**
111		110
111	and	110
110		101
		011

as being the only partitions making a contribution. The result of section 19 follows at once.

34. **The Case of a Normal Universe.** A normal universe is characterized by the relationship that $\lambda_r = 0$ when $r > 2$. It follows that it is only necessary to compute the coefficients of those partitions giving powers of λ_2.

Wishart (5) (7) has developed the partition analysis of the k function in the case of a normal parent while Georgescu has studied the corresponding m function. It is not the purpose of this section to make extensive study of the case of the normal parent but simply to indicate that the results of section 24 are immediately applicable. As an illustration we write the values of $\lambda_1(f_2)$, $\lambda_2(f_2)$, $\lambda_3(f_2)$ and $\lambda_4(f_2)$ in the case of a normal universe. The terms are given successively, by

1	**2**	**8**	**48**
2	11	110	1100
	11	011	0110
		101	0011
			1001

and hence

$$\lambda_1(f_2) = b_2 n \lambda_2$$

$$\lambda_2(f_2) = 2[b_2^2 n + b_{11}^2 n(n-1)]\lambda_2^2$$

$$\lambda_3(f_2) = 8[b_2^3 n + 3b_2 b_{11}^2 n(n-1) + b_{11}^3 n(n-1)(n-2)]\lambda_2^3$$

$$\lambda_4(f_2) = 48[b_2^4 n + 6b_2^2 b_{11}^2 n(n-1) + b_{11}^4 n(n-1) + 4b_2 b_{11}^3 n(n-1)(n-2) + 2b_{11}^4 n(n-1)(n-2) + b_{11}^4 n(n-1)(n-2)(n-3)]\lambda_2^4.$$

It is only necessary to substitute the b's to obtain the results for different values of f. This is done in Table IV.

TABLE IV

The first four Thiele moments of f_2 for various sample functions in the case of a normal universe

Sample function	$\lambda_1(f_2)$	$\lambda_2(f_2)$	$\lambda_3(f_2)$	$\lambda_4(f_2)$
m_2	$\frac{(n-1)}{n}\lambda_2$	$\frac{2(n-1)}{n^2}\lambda_2^2$	$\frac{8(n-1)}{n^3}\lambda_2^3$	$\frac{48(n-1)\lambda_2^4}{n^4}$
k_2	λ_2	$\frac{2\lambda_2^2}{n-1}$	$\frac{8\lambda_2^3}{(n-1)^2}$	$\frac{48\lambda_2^4}{(n-1)^3}$
l_2	$\frac{(n-1)}{n}\lambda_2$	$\frac{2(n-1)}{n^2}\lambda_2^2$	$\frac{8(n-1)\lambda_2^3}{n^3}$	$\frac{48(n-1)\lambda_2^4}{n^4}$
m_2'	λ_2	$\frac{2\lambda_2^2}{n}$	$\frac{8\lambda_2^3}{n^3}$	$\frac{48\lambda_2^4}{n^4}$
h_2	λ_2	$\frac{2\lambda_2^2}{n-1}$	$\frac{8\lambda_2^3}{(n-1)^2}$	$\frac{48\lambda_2^4}{(n-1)^3}$
h_2'	0	$\frac{2\lambda_2^2}{n(n-1)}$	$\frac{8(n-2)\lambda_2^3}{n^2(n-1)^2}$	$\frac{48(n^2-3n+3)\lambda_2^4}{n^3(n-1)^3}$

One surmises that the general value of

$$\lambda_r(f_2) \text{ is } 2^{r-1}(r-1)!\,\lambda_2^r B\colon \begin{array}{l} 11000\cdots 0 \\ 01100\cdots 0 \\ 00110\cdots 0 \\ \cdots\cdots\cdots \\ 00000\cdots 11 \\ 10000\cdots 01 \end{array}$$

where B represents the b coefficient of the r rowed partition. This induction appears consistent with the fact that

$$\lambda_{r+1}(k_2) = \frac{2^r r!\, \lambda_2^{r+1}}{(n-1)^r}$$

as shown by John Wishart (7). The whole subject of the Thiele moments of the general function in the case of a normal universe would make an interesting subject of investigation.

35. **Summary and Conclusion.** The contributions of this paper include

1. The definitions of specific moment functions in terms of power sums.
2. The use of indeterminate multipliers in representing a general isobaric moment function.
3. The finding of the expected value of products of these functions by algebraic methods.
4. The use of tables in writing these expected values in terms of moments (or of moments about a fixed point) of the universe.
5. The finding of the expected values of specific moment functions by substitution.
6. Means of establishing the expansion of new moment functions which are defined by their expected values.
7. The introduction of the sample function of weight r whose expected value is μ_r.
8. The introduction of the sample function of weight r whose expected value is $\mu_1'^r$.
9. The two way partition formulae of weight $\leqq 8$ which do not involve unit parts.

The use of these partition formulae in writing:

10. The moments about a fixed point of f_r in terms of moments.
11. The moments of f_r in terms of moments.
12. The Thiele moments of f_r in terms of moments.
13. The moments about a fixed point of f_r in terms of Thiele moments.
14. The moments of f_r in terms of Thiele moments.
15. The Thiele moments of f_r in terms of Thiele moments.
16. Special rules in the case of Thiele moments.
17. The applicability of these results to a given sample moment function and hence the derivation of varied results, of such authors as Thiele, Tchouproff, Church, Fisher, Craig, and Georgescu, from the same partition formulae.
18. The simplicity of the formulae when h_r' is used as the sample function.
19. The application of the synthetic formulae to the Craig method.
20. The applicability of the theory to a normal universe.

The introduction of such general procedure opens up a wide field for future study. It is impossible in a single paper dealing with so broad a subject to do more than to outline the general scheme by which two way partitions can be

used as a central formulization of the various formulae for moments of moments. More detailed proofs and more extensive analysis of the more important of the special cases will undoubtedly be supplied by later writers.

In later papers the author will show how the partition representation can be used in the case of multivariate distributions and how it can also be used, in connection with the sampling polynomials introduced by H. C. Carver (11), to represent the more complex formulae obtained in the case of finite sampling.

It is obvious that the author is indebted to the classical moment studies of Fisher and Craig. He also wishes to acknowledge his indebtedness to Prof. Craig and to Prof. Carver who have read the manuscript and have made valuable suggestions.

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NOTES

A COEFFICIENT OF CORRELATION BETWEEN SCHOLARSHIP AND SALARIES

INTRODUCTION

Some might doubt that it is correct to apply a coefficient of correlation to show the relationship between scholarship and salaries. This coefficient can be trusted to give at least a rough approximation, which is all that is necessary in the inexact science of vocation. It is fictitious accuracy to be too finical in the application of formulas. Therefore, a coefficient of correlation between scholarship and salaries is a valuable part of human knowledge.

Would it be worth while to find this coefficient if it is based upon the experience of the American Telegraph and Telephone Company? Since the employment practices of this company are not representative of the employment practices of business at large, one might doubt the validity of drawing general conclusions from such specialized data. The coefficient for business at large is probably less than the coefficient for the Bell System; the value of this knowledge is enhanced if we know the latter coefficient. Since this company is very large, a coefficient between scholarship and salaries would be valuable, even if this coefficient applies only to the Bell System and to other companies having approximately the same employment practices.

An article[1] by Mr. Walter S. Gifford, President of the Bell System, contains a discussion of some of the relationships between scholarship and salaries. President Gifford, however, did not determine in the case of the Bell System a coefficient of correlation between scholarship and salaries.

The purpose of this article is not a new contribution to statistical method, but is an application of the method[2] of finding the coefficient of correlation when the two variables have not been quantitatively measured. This method will be applied to the chart on page 672 of President Gifford's article, in order to determine for the Bell System the coefficient of correlation between scholarship and salaries.

FINDING THE COEFFICIENT OF CORRELATION

An explanation of the chart. It is based on the experience of 2,144 Bell System employees over five years out of college. First, assume these employees

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[1] It is entitled "Does Business Want Scholars?" and was printed in the May 1928 issue of Harper's Magazine.

[2] It can be found in Elderton's "Frequency Curves and Correlation."

are grouped according to their grades in college. In the high scholarship group put those who graduated in the highest third of their classes. The middle and low scholarship groups are formed in like manner. Secondly, suppose the same employees are divided into three equal groups according to their salaries. Then, the salary of any one of the employees would be high, middle, or low.

Assume a hypothetical group of 300 employees who are college graduates. Suppose that the scholarship of 100 of them was high, that the scholarship of 100 of them was middle, and that the scholarship of the others was low. Also assume that the salary experience of these 300 employees is the same as that of the 2,144 employees of the Bell System.

The 300 employees can be grouped according to the following table.

TABLE NO. 1

Salary	Scholarship			Totals
	Low	Middle	High	
High	22	24	48	94
Middle	31	39	27	97
Low	47	37	25	109
Totals	100	100	100	300

This table can be combined as follows.

TABLE NO. 2

Salary	Scholarship	
	Low & Middle	High
High	c	d
Middle & Low	a	b

Then, $c = 46$, $a = 154$, $d = 48$, and $b = 52$. Assume $N = 300$.

Assume x is a function of grades received in college. Suppose y is a function of salaries received. Assume that the frequencies x and y both follow the normal curve of error whose standard deviation is equal to one. Also assume that the average of x and the average of y are both equal to zero. It is a matter of common knowledge that salaries are not arranged in a symmetrical fashion; y is not a linear function of salaries.

In the formulas which follow, r is the symbol for the coefficient of correlation. These formulas are applied to Table No. 2. We have

$$\frac{1}{\sqrt{2\pi}} \int_0^h e^{-\frac{1}{2}x^2}\, dx = \frac{(a + c) - (b + d)}{2N} = .167, \text{ and } h = .4316.$$

Also

$$\frac{1}{\sqrt{2\pi}}\int_0^k e^{-\frac{1}{2}y^2}\,dy = \frac{(a+b)-(c+d)}{2N} = .187, \text{ and } k = .4874.$$

Then,

$$H = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}h^2} = .3635, \text{ and } K = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}k^2} = .3543.$$

All the quantities except r in the following approximate equation are known:

$$\frac{ad-bc}{N^2HK} = r + \frac{r^2}{2}hk + \frac{r^3}{6}(h^2-1)(k^2-1) + \frac{r^4}{24}h(h^2-3)k(k^2-3) + \frac{r^5}{125}(h^4-6h^2+3)(k^4-6k^2+3).$$

Therefore,

$$.0261r^5 + .0681r^4 + .1034r^3 + .1052r^2 + r - .4314 = 0.$$

Then, r is approximately equal to .4051. Consequently, for practical purposes we can assume that $r = .4$.

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NOTE ON THE DERIVATION OF THE MULTIPLE CORRELATION COEFFICIENT

Consider N observed values of each of n variables. These $n \cdot N$ values may be tabulated in a double-entry table as follows:

$$\begin{array}{l} X_{11}\ X_{12}\ X_{13} \cdots X_{1N} \\ X_{21}\ X_{22}\ X_{23} \cdots X_{2N} \\ \cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots \\ X_{n1}\ X_{n2}\ X_{n3} \cdots X_{nN} \end{array}$$

where X_{ik} is the k^{th} value of the i^{th} variable.

Using the i^{th} variable as the dependent variable, the general linear relationship between the n variables may be expressed by

$$x_i = {}_ia_1\,x_1 + {}_ia_2\,x_2 + \cdots + {}_ia_{i-1}\,x_{i-1} + {}_ia_{i+1}\,x_{i+1} + \cdots + {}_ia_n\,x_n \qquad (1)$$

where

${}_ia_j$ is the general parameter which is to be determined empirically;

$x_j = X_j - M_j$;

M_j is the arithmetic mean of the j^{th} variable.

By the method of least squares, the constants of (1) must satisfy the normal equations:

$$(\Sigma x_1^2)_i a_1 + (\Sigma x_1 x_2)_i a_2 + \cdots + (\Sigma x_1 x_{i-1})_i a_{i-1} + (\Sigma x_1 x_{i+1})_i a_{i+1} + \cdots + (\Sigma x_1 x_n)_i a_n = \Sigma x_1 x_i$$

$$(\Sigma x_2 x_1)_i a_1 + (\Sigma x_2^2)_i a_2 + \cdots + (\Sigma x_2 x_{i-1})_i a_{i-1} + (\Sigma x_2 x_{i+1})_i a_{i+1} + \cdots + (\Sigma x_2 x_n)_i a_n = \Sigma x_2 x_i$$

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$$(\Sigma x_{i-1} x_1)_i a_1 + (\Sigma x_{i-1} x_2)_i a_2 + \cdots + (\Sigma x_{i-1} x_n)_i a_n = \Sigma x_{i-1} x_i$$

$$(\Sigma x_{i+1} x_1)_i a_1 + (\Sigma x_{i+1} x_2)_i a_2 + \cdots + (\Sigma x_{i+1} x_n)_i a_n = \Sigma x_{i+1} x_i$$

. . . . . .

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$$(\Sigma x_n x_1)_i a_1 + (\Sigma x_n x_2)_i a_2 + \cdots + (\Sigma x_n^2)_i a_n = \Sigma x_n x_i$$

where

$$(\sum x_i x_j) = \sum_{k=1}^{N} (X_{ik} - M_i)(X_{jk} - M_j).$$

But

$$(\Sigma x_i x_j) = N r_{ij} \sigma_i \sigma_j,$$

$$(\Sigma x_i^2) = N \sigma_i^2 = N r_{ii} \sigma_i \sigma_i \tag{2}$$

where

r_{ij} is the Pearsonian coefficient of correlation between the i^{th} and j^{th} variables,

σ_i, the standard deviation of the i^{th} variable.

Substituting the right members of (2) in the normal equations, we obtain the system:

$$\begin{aligned}
&\sum_{k=1}^{n} r_{1k} \sigma_1 \sigma_k \, {}_i a_k = 0 \\
&\sum_{k=1}^{n} r_{2k} \sigma_2 \sigma_k \, {}_i a_k = 0 \\
&\qquad \cdot \quad \cdot \\
&\qquad \cdot \quad \cdot \\
&\sum_{k=1}^{n} r_{i-1,k} \, \sigma_{i-1} \sigma_k \, {}_i a_k = 0 \\
&\sum_{k=1}^{n} r_{i+1,k} \, \sigma_{i+1} \sigma_k \, {}_i a_k = 0 \\
&\qquad \cdot \quad \cdot \\
&\qquad \cdot \quad \cdot \\
&\sum_{k=1}^{n} r_{nk} \sigma_n \sigma \, {}_i a_k = 0
\end{aligned} \tag{3}$$

where

$$ {}_{i}a_{i} = -1. $$

Let

$$ A = \begin{vmatrix} r_{11}\sigma_1\sigma_1 & \cdots & r_{n1}\sigma_n\sigma_1 \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ r_{1n}\sigma_1\sigma_n & & r_{nn}\sigma_n\sigma_n \end{vmatrix}, \tag{4} $$

A_{ij} be the first minor of the element $r_{ij}\sigma_i\sigma_j$ in A, ${}_{ik}A$ be A with the i^{th} and k^{th} columns interchanged, and ${}_{ik}A_{ii}$ be the first minor of the element in the i^{th} column and i^{th} row of ${}_{ik}A$.

Solving (3) for ${}_{i}a_{k}$ by Cramer's rule, we find

$$ {}_{i}a_{k} = \frac{{}_{ik}A_{ii}}{A_{ii}}. $$

But it can easily be proved that

$$ {}_{ik}A_{ii} = (-1)^{i-k+1}A_{ik}; $$

hence

$$ {}_{i}a_{k} = (-1)^{i-k+1}\frac{A_{ik}}{A_{ii}}. $$

Using cofactors of A instead of minors, we have

$$ {}_{i}a_{k} = (-1)^{i-k+1}\frac{(-1)^{i+k}D_{ik}}{D_{ii}} = -\frac{D_{ik}}{D_{ii}}. $$

Without writing the determinant out in full, we notice that the σ's can be factored out. Hence

$$ \begin{aligned} {}_{i}a_{k} &= -\frac{\sigma_1^2\sigma_2^2\cdots\sigma_{k-1}^2\sigma_k\sigma_{k+1}^2\cdots\sigma_{i-1}^2\sigma_i\sigma_{i+1}^2\cdots\sigma_n^2 K_{ik}}{\sigma_1^2\sigma_2^2\cdots\sigma_{i-1}^2\sigma_{i+1}^2\cdots\sigma_n^2 K_{ii}} \\ &= -\frac{\sigma_i K_{ik}}{\sigma_k K_{ii}}, \end{aligned} \tag{5} $$

where

$$ K = \begin{vmatrix} r_{11} & \cdots & r_{1n} \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ r_{n1} & & r_{nn} \end{vmatrix}. $$

Using these derived values for the coefficients, we may write (1) in the symmetric form:

$$\frac{K_{i1}}{\sigma_1}(X_1 - M_1) + \frac{K_{i2}}{\sigma_2}(X_2 - M_2) + \cdots + \frac{K_{in}}{\sigma_n}(X_n - M_n) = 0,$$

or

$$\sum_{j=1}^{n} \frac{K_{ij}x_j}{\sigma_j} = 0. \tag{6}$$

For a multiple correlation coefficient, we use the formula

$$R_i^2 = 1 - \frac{\sum_{j=1}^{N}\left[x_{ij} - \left(\sum_{k=1}^{i-1} {}_ia_k x_{kj} + \sum_{k=i+1}^{n} {}_ia_k x_{kj}\right)\right]^2}{N\,\sigma_i^2}$$

which measures the amount of observed dispersion from the regression plane in which X_i is the dependent variable.

Substituting the values for the a's, we find

$$R_i^2 = 1 - \frac{\sum_{j=1}^{N}\left(\frac{K_{i1}x_{1j}}{\sigma_1} + \frac{K_{i2}x_{2j}}{\sigma_2} + \cdots + \frac{K_{in}x_{nj}}{\sigma_n}\right)^2}{K_{ii}^2\,N}.$$

Squaring the bracket expression and using (2), we obtain

$$R_i^2 = 1 - \frac{1}{K_{ii}^2}\left[\sum_{k=1}^{N}\sum_{l=1}^{N}\left(\frac{K_{ik}K_{il}\sum_{j=1}^{N} x_{kj}x_{lj}}{N\sigma_k\sigma_l}\right)\right]$$

$$= 1 - \frac{1}{K_{ii}^2}\left[\sum_{k=1}^{n}\sum_{l=1}^{n} K_{ik}K_{il}r_{kl}\right]$$

$$= 1 - \frac{1}{K_{ii}^2}\left[\sum_{k=1}^{n}\left(K_{ik}\sum_{l=1}^{n} K_{il}r_{kl}\right)\right].$$

The second sum is the sum of the products of the elements in the k^{th} row by the cofactors of the elements in the i^{th} row. This sum is necessarily zero unless $k = i$; but if $k = i$, this sum is equal to K.

$$R_i^2 = 1 - \frac{1}{K_{ii}^2}(K_{ii}\,K) = 1 - \frac{K}{K_{ii}}.$$

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NOTE ON NUMERICAL EVALUATION OF DOUBLE SERIES[1]

1. The Euler-Maclaurin summation formula has been extended to two variables by Dr. Sheppard,[2] and Mr. Irwin,[3] to determine cubature formulas. A more complicated two-dimensional form was given by Baten[4] involving product polynomials, for which a remainder term was also calculated. The purpose of this note is to apply the simpler formula to the numerical evaluation of double series of positive terms. The method may be extended to multiple series of order $p > 2$. If the double series converges one may sum by rows (or columns), using the ordinary sum formula twice. The method is to take out a rectangular block of mn terms and then apply the formula to the remaining terms. By taking m and n sufficiently large one may cause the series resulting from the formula to converge sufficiently rapidly to obtain the sum to the desired number of decimal places. For practical work the error may be estimated because of the asymptotic character of the series involved in the Euler-Maclaurin formula.

Write this in the form

$$\sum_a^{s-1} f(x) = \int_a^s f(x)dx + \tfrac{1}{2}f(a) - \tfrac{1}{2}f(s) - \frac{f'(a) - f'(s)}{12} + \frac{f'''(a) - f'''(s)}{720} - \frac{f^{V}(a) - f^{V}(s)}{30240} + \frac{f^{VII}(a) - f^{VII}(s)}{1209600} - \cdots + (-1)^r B_r \frac{f^{(2r-1)}(a) - f^{(2r-1)}(s)}{(2r)!} + \cdots. \tag{1}$$

If $s \to \infty$ one has accordingly in the ordinary case of convergence

$$\sum_a^{\infty} f(x) = \int_a^{\infty} f(x)dx + \tfrac{1}{2}f(a) - \frac{f'(a)}{12} + \frac{f'''(a)}{720} - \frac{f^{V}(a)}{30240} + \cdots. \tag{2}$$

Now define $v(x) = \sum_{y=b}^{\infty} u(x, y) = \int_b^{\infty} u(x, y)\,dy + \cdot\tfrac{1}{2}u(x, b) - \frac{u_y(x, b)}{12} + \frac{u_{y^3}(x, b)}{720} - \cdots$ and $w(y) = \sum_{x=a}^{\infty} u(x, y) = \int_a^{\infty} u(x, y)dx + \tfrac{1}{2}u(a, y) - \frac{u_x(a, y)}{12} + \frac{u_{x^3}(a, y)}{720} - \cdots$, then

$$\sum_{x=1}^{\infty}\sum_{y=1}^{\infty} u(x, y) = \sum_{x=1}^{a-1}\sum_{y=1}^{b-1} u(x, y) + \sum_{x=1}^{\infty} v(x) + \sum_{y=1}^{b-1} w(y)$$

$$= \int_1^{\infty} v(x)dx + \tfrac{1}{2}v(1) - \frac{v'(1)}{12} + \frac{v'''(1)}{720} - \frac{v^{V}(1)}{30240} + \cdots + \sum_{x=1}^{a-1}\sum_{y=1}^{b-1} u(x, y) + \int_1^b w(y)dy - \tfrac{1}{2}w(b) + \tfrac{1}{2}w(1) + \frac{w'(b) - w'(1)}{12} - \frac{w'''(b) - w'''(1)}{720} + \cdots. \tag{3}$$

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[1] Presented to the Society, Nov. 30, 1934.

[2] W. F. Sheppard, "Some Quadrature Formulae," Proc. London Math. Soc., Vol. xxxii, 1900.

[3] J. O. Irwin, "Tracts for Computers," No. X, Cambridge Univ. Press, 1923, On Quadrature and Cubature.

[4] W. D. Baten, "A Remainder for the Euler-Maclaurin summation formula in two independent variables," Amer. Journal of Math., Vol. 54, 1932, pp. 265–275.

Instead of this one may use $\sum_{x=1}^{a-1}\sum_{y=1}^{b-1} u(x, y) + \sum_{y=1}^{\infty} w(y) + \sum_{x=1}^{a-1} v(x)$. The scheme of the double series may be illustrated by a sketch of a quadrant of the xy-plane in which the point (x, y) represents the term $u(x, y)$.

Evidently by taking a combination of results from (3) one may evaluate quite readily such finite sums as $\sum_{x=p}^{q}\sum_{y=r}^{t} u(x, y)$ where q and t are large.

As an illustration of (3) consider $\sum\sum (x^2 + a^2y^2)^{-2}$. Here one needs to evaluate the integral of the summand. The transformations $x = ay \tan \theta$ and $y = 1/t$ lead to a form which may be integrated by parts. The more complicated form $\sum\sum (ax^2 + 2bxy + cy^2)^{-s}$ for the case in which $s > 3/2$, $a > 1$, might be handled by using $x = 1/t$ and approximate integration by Simpson's rule.

Take as a second example $\sum\sum (x + y)^{-p}$, $p > 2$. The case of $p = 4$ was carried out by taking $a = b = 10$ in (3) and carrying the computation to twelve decimals. The series involved converge rapidly and a result was obtained which differed by 2 in the 12th place from the true value $0.119\ 733\ 669\ 448^{+}$. By summing diagonally one may convert this to the simple series $\sum_{1}^{\infty} z(z + 1)^{-4}$ or $\sum_{2}^{\infty} (s - 1)s^{-4} = \sum_{1}^{\infty} (s^{-3} - s^{-4})$. The method of summation diagonally may be extended to $\sum\sum (x + ay)^{-p}$, $p > 2$, $a > 0$, by the applications of the Euler-Maclaurin sum formulas (1), (2) in succession after a triangular array of terms have been omitted.

The form $\sum\sum x^{-p}y^{-q}$ can be written as the product of the single series $(\sum x^{-p})(\sum y^{-q})$.

2. Another method of numerical evaluation is the analog of that used for single series by the author.[5] Instead of rectangles one has right prisms of square or rectangular cross-section. Instead of shifting the rectangles one unit to the right to determine upper and lower bounds the prisms are shifted diagonally so that they go effectively one unit in each variable. In the case of a square base each prism is moved along the 45° line one diagonal unit length. For the lower bound instead of trapezoids one uses truncated prisms. For example, the prism of height u_{mn} is cut by two planes, one determined by the upper vertices u_{mn}, $u_{m,n+1}$, $u_{m+1,n}$ and the other by the upper vertices $u_{m+1,n}$, $u_{m,n+1}$, $u_{m+1,n+1}$ of the truncated prism. The surface $z = u(m, n)$ passes through all the upper corners of the truncated prisms. Each prism is composed of two truncated triangular prisms. Now the volume of such a triangular prism is the arithmetic mean of its vertical edges multiplied by the area of its base.

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[5] "A New Method for Finding the Numerical Sum of an Infinite Series," Amer. Math. Monthly, vol. XL, No. 9, Nov., 1933, pp. 537–542.

Hence the difference in volume between the truncated rectangular prism mentioned above and the prism of uniform hight $z = u_{mn}$ can be shown to be

$$(4) \qquad (5u_{mn} - u_{m+1,\,n+1} - 2u_{m+1,\,n} - 2u_{m,\,n+1})/6.$$

Let us consider series whose corresponding surfaces do not rise above these truncated prisms. This sort of truncated prism differs less from the volume under the surface than the one formed by the diagonal joining the other pair of upper vertices and planes through it for upper faces. The lower bound for the remainder is the volume under the surface extending to infinity in the m and n directions plus the sum of these differences. Accordingly one determines as the lower bound for the remainder $R_{m-1,\,n-1}$ after summing a rectangular array $\sum_{i=1}^{m-1}\sum_{j=1}^{n-1} u_{i,\,j}$ the form

$$(5) \qquad \begin{aligned}&(2u_{m,1} + 2u_{1,n} - 5u_{m,n})/6 + \tfrac{1}{2}\sum_{i=1}^{\infty} u_{m+i,1} + \tfrac{1}{2}\sum_{j=1}^{\infty} u_{1,n+j} + \tfrac{1}{2}\sum_{i=1}^{m} u_{i,n} \\ &\quad + \tfrac{1}{2}\sum_{j=1}^{n} u_{m,j} + \int_1^{\infty}\int_m^{\infty} u_{m,n}\,dmdn + \int_n^{\infty}\int_1^{m} u_{m,n}\,dmdn < R_{m-1,n-1}.\end{aligned}$$

The upper bound may likewise be given as follows:

$$(6) \qquad R_{m-1,n-1} < S + T + \int_{n-1}^{\infty}\int_{m-1}^{\infty} u_{m,n}\,dmdn - k\left[\sum_{j=n-1}^{\infty} u_{m-1,j} + \sum_{i=m}^{\infty} u_{i,n-1}\right]$$

where

$$(7) \qquad S = \sum_{i=m}^{\infty}\sum_{j=1}^{n-1} u_{ij}, \qquad T = \sum_{j=n}^{\infty}\sum_{i=1}^{m-1} u_{ij},$$

$$(8) \qquad k = \frac{\displaystyle\int_{n-1}^{\infty}\int_{m-1}^{\infty} u_{m,n}\,dmdn - \int_n^{\infty}\int_m^{\infty} u_{m,n}\,dmdn - \sum_{j=n-1}^{\infty} u_{m-1,j} - \sum_{i=m}^{\infty} u_{i,m-1}}{u_{m-1,n-1} + u_{m,n-1}}.$$

An alternate definition of k is

$$(9) \qquad k' = \left[\int_{n-1}^{n}\int_{m-1}^{m} u_{m,n}\,dmdn - u_{m,n}\right] \div (u_{m-1,n-1} - u_{m,n}).$$

An illustration is afforded by $\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} (m+1)^{-4}$ for which $k = .45614$, $k' = .44536$ when $m = n = 10$ in (8), (9). In this case (5) gave an error of -14×10^{-6} and (6) an error of 10^{-6}.

S and T may be evaluated by the method published in the Monthly.[6]

One must assume that k increases with m and n. It is evident that for this

[6] Loc. cit.

method and for the one in the Monthly differentiability is not required but only integrability, conditions less restrictive than those required by the Euler-Maclaurin summation formulas. It is also clear that the method may be extended to multiple series of positive terms of multiplicity greater than two.

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REPORT OF THE ANNUAL MEETING OF THE INSTITUTE OF MATHEMATICAL STATISTICS

The meeting of the Institute of Mathematical Statistics for 1936 was held in Chicago on December 28–30 in connection with the meetings of the American Statistical Association and the Econometric Society.

In addition to the sessions at which voluntary papers were read, a session with invited papers was held on the morning of December 30. At the invitation of the Program Committee, Professor P. R. Rider presented a paper on "Recent Advances in Mathematical Statistics: Factorial Design" and Professor Harold Hotelling spoke on "The Analysis of Sets of Correlated Variates."

Professor C. C. Craig of the University of Michigan and Professor A. R. Crathorne of the University of Illinois constituted the Program Committee.

At the business meeting of the Institute, the following officers were elected for the year 1937: President, Dr. W. A. Shewhart; Vice-Presidents, Professors P. R. Rider and B. H. Camp; Secretary-Treasurer, Professor A. T. Craig.

The Institute voted that it would presumably hold its 1937 meeting with the American Mathematical Society.

ALLEN T. CRAIG,
Secretary.

NOTICE TO SUBSCRIBERS

Plans are under way to include in the ANNALS a new section, entitled "Numerical Illustrations of Statistical Methodology." This new section will be a regular feature of the ANNALS, and will deal with the application of statistical technique and theory to the solution of problems in various fields. It is hoped that this new section will be of considerable value to those who are primarily interested in numerical applications of the more recent theoretical developments in mathematical statistics.

The Editor will welcome contributions to this new section of the ANNALS.